Mathematical Analysis for a SEIS Typhoid Epidemic Model

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Abstract
In the present paper, We discussed a Susceptible-Exposed-Infected-Susceptible typhoid fever mathematical model. Obtained disease-free and endemic equilibria and basic reproduction number R_0 for the model. Local and global stability studied for both equilibrium points using Routh–Hurwitz criteria ,Castillo-Chavez criteria and Lyapunov-Lasalle principle respectively . Case study also be done numerically and graphically which showed that how the disease will be dies out and when the disease will persists.

Keywords: Typhoid , reproduction number, Routh–Hurwitz criteria ,Castillo-Chavez criteria , Lyapunov-Lasalle principle.

Mathematics Subject Classification : 34D20,49J15,92D25,92D30,93D20.

1. Introduction

Typhoid fever is caused by Salmonella typhi bacteria having by contaminated food and water or by close contact with infected person. Due to this high fever, headache, abdominal pain and either constipation or diarrhea just like this symptoms are seeing in humans. Castillo-Chavez et al. gave reproductive number and stability for some epidemic models and a theorem on globally analysis of equilibrium points known as Castillo-Chavez criterion[1].Karunditu J.W. et al. formulated a deterministic typhoid mathematical model SEIRS incorporating exposed humans and studied stability using Castillo-Chavez method [2]. Muhammad A.K. et al. [3] considered a mathematical model SEIR to understand the dynamics of this disease. In [4] , Mushayabasa S. has formulated and analyzed a mathematical model on controlling the spread of the typhoid disease through the basic reproductive number and numerical simulations . Nithri J.K. et al. have been formulated a mathematical model PSIT to study the dynamics of typhoid fever disease incorporating safety against infection [5].

2. Mathemaical Analysis of the Model

We have referred a typhoid mathematical model SEIRS of Karunditu J.W. et al. [2] and modified it by adding transmission rates ηI .So, the new model obtained as SEIS typhoid transmission model. In the model, human population is assumed as susceptible (S) , Exposed (E) , infectious (I) and recovered (R) individuals at time t . Assume the total number of population at time t is $N(t) = S(t) + E(t) + I(t) + R(t)$.

The following figure 1 represents the flow of individuals for the considered model :

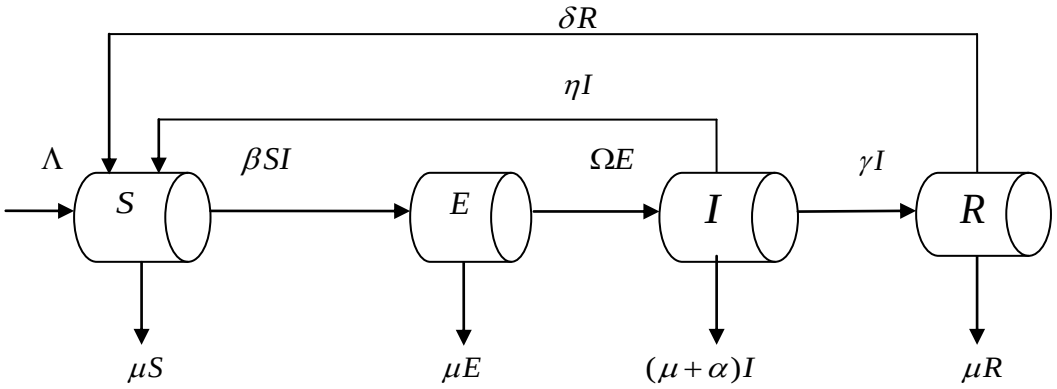


Figure 1 : SEIS Typhoid Transmission model

where the symbols stand for

Λ	Human recruitment rate (birth)
μ	Natural death rate
α	Disease induced death rate
β	Disease interaction rate
Ω	Unprotected symptoms showing rate
γ	Infective recovery rate
δ	Transmission rate from R to S
η	Transmission rate from I to S

All parameters are assumed nonnegative .

The differential equations for the model are

$$\frac{dS}{dt} = \Lambda + \delta R + \eta I - \mu S - \beta SI$$
$$\frac{dE}{dt} = \beta SI - (\mu + \Omega)E \qquad (1)$$
$$\frac{dI}{dt} = \Omega E - (\mu + \alpha + \eta + \gamma)I$$
$$\frac{dR}{dt} = \gamma I - (\delta + \mu)R$$

Consider ,the feasible region D for system (1) is

$$D = \{(S, E, I, R) \in ¡^4_+ : S \geq 0, E \geq 0, I \geq 0, R \geq 0, 0 \leq S + E + I + R \leq \frac{\Lambda}{\mu}\}.$$

3. Equilibrium Points and Reproduction number

3.1 Disease-free equilibrium (DFE) (E_1)

From system (1) , we have

$$\Lambda + \delta R + \eta I - \mu S - \beta SI = 0$$

$$\beta SI-(\mu+\Omega)E=0 \qquad (2)$$

$$\Omega E-(\mu+\alpha+\eta+\gamma)I=0$$

$$\gamma I-(\delta+\mu)R=0$$

Assume that if $I=0$ (no disease),then on solving all equations of system (2) ,we have

$S=\frac{\Lambda}{\mu}$, $E=0$, $R=0$.

Thus, $E_1=(\frac{\Lambda}{\mu},0,0,0)$ is DFE.

3.2Endemic equilibrium (EE)(E_2)

Again, the system (1) becomes system (3) such that

$$\Lambda+\delta R^*+\eta I^*-\mu S^*-\beta S^* I^*=0$$

$$\beta S^* I^*-(\mu+\Omega)E^*=0 \qquad (3)$$

$$\Omega E^*-(\mu+\alpha+\eta+\gamma)I^*=0$$

$$\gamma I^*-(\delta+\mu)R^*=0$$

From third, fourth and second equations ,respectively, we obtain

$$E^*=\left(\frac{\mu+\alpha+\gamma+\eta}{\Omega}\right)I^*,\ R^*=\frac{\gamma}{(\delta+\mu)}I^*, S^*=\frac{(\Omega+\mu)(\mu+\alpha+\gamma+\eta)}{\Omega\beta}.$$

Substitute all these values in equation first , then we get

$$I^*=\frac{(\delta+\mu)[\Omega\beta\Lambda-\mu(\Omega+\mu)(\mu+\alpha+\gamma+\eta)]}{\beta[(\delta+\mu)(\Omega+\mu)(\mu+\alpha+\gamma+\eta)-\{\eta(\delta+\mu)+\delta\gamma\}\Omega]}$$

Using the value of I^* , finally we obtain the endemic equilibrium point $E_2=(S^*,E^*,I^*,R^*)$ such that

$$S^*=\frac{(\Omega+\mu)(\mu+\alpha+\gamma+\eta)}{\Omega\beta},\ E^*=\frac{(\mu+\alpha+\gamma+\eta)(\delta+\mu)[\Omega\beta\Lambda-\mu(\Omega+\mu)(\mu+\alpha+\gamma+\eta)]}{\Omega\beta[(\delta+\mu)(\Omega+\mu)(\mu+\alpha+\gamma+\eta)-\{\eta(\delta+\mu)+\delta\gamma\}\Omega]}$$

,

$$I^*=\frac{(\delta+\mu)[\Omega\beta\Lambda-\mu(\Omega+\mu)(\mu+\alpha+\gamma+\eta)]}{\beta[(\delta+\mu)(\Omega+\mu)(\mu+\alpha+\gamma+\eta)-\{\eta(\delta+\mu)+\delta\gamma\}\Omega]},$$

$$R^*=\frac{\gamma[\Omega\beta\Lambda-\mu(\Omega+\mu)(\mu+\alpha+\gamma+\eta)]}{\beta[(\delta+\mu)(\Omega+\mu)(\mu+\alpha+\gamma+\eta)-\{\eta(\delta+\mu)+\delta\gamma\}\Omega]}.$$

3.3 Reproduction Number R_0

We employ next generation matrix method to find out R_0 .

Let $Y = (E, I)^T$. Then system (1) can be written as

$$\frac{dY}{dt} = M(Y) - U(Y)$$

where , $M(Y) = \begin{bmatrix} \beta SI \\ 0 \end{bmatrix}, U(Y) = \begin{bmatrix} (\mu + \Omega) E \\ (\mu + \gamma + \eta + \alpha) I - \Omega E \end{bmatrix}$.

The Jacobian matrices of $M(Y)$ and $U(Y)$ at the disease free equilibrium E_1 are, respectively,

$$DM(E_1) = \begin{bmatrix} 0 & \beta S \\ 0 & 0 \end{bmatrix}, DU(E_1) = \begin{bmatrix} (\mu + \Omega) & 0 \\ -\Omega & (\mu + \gamma + \eta + \alpha) \end{bmatrix}.$$

Then,

$$MU^{-1} = \begin{bmatrix} \frac{\Omega \beta S}{(\mu + \Omega)(\alpha + \mu + \gamma + \eta)} & \frac{\beta S}{(\alpha + \mu + \gamma + \eta)} \\ 0 & 0 \end{bmatrix}$$

is the next generation matrix of system (1).

Now, $\left| MU^{-1} - \lambda I \right| = 0$

$\begin{vmatrix} \frac{\Omega \beta S}{(\mu + \Omega)(\alpha + \mu + \gamma + \eta)} - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$ which implies that $\lambda_1 = 0$, $\lambda_2 = \frac{\Omega \beta S}{(\mu + \Omega)(\alpha + \mu + \gamma + \eta)}$.

So, the basic reproduction number at disease free equilibrium E_1 is given by

$$R_0 = \frac{\Omega \beta \Lambda}{\mu(\mu + \Omega)(\alpha + \mu + \gamma + \eta)} .$$

4. Stability Analysis

(A) Local stability of disease-free equilibrium

Theorem 1 . If $R_0 < 1$,then the DFE is locally asymptotically stable(LAS) .

Proof . Consider, the matrix at disease-free equilibrium points will be

$$J(E_1) = \begin{bmatrix} -\mu & 0 & -\beta S_0 & \delta \\ 0 & -(\Omega + \mu) & \beta S_0 & 0 \\ 0 & \Omega & -(\mu + \alpha + \gamma + \eta) & 0 \\ 0 & 0 & \gamma & -(\delta + \mu) \end{bmatrix}.$$

Now,

$$\begin{aligned} trace(J(E_1)) &= -\mu - (\Omega + \mu) - (\mu + \alpha + \gamma + \eta) - (\delta + \mu) \\ &= -(4\mu + \Omega + \alpha + \gamma + \eta + \delta) < 0 \end{aligned}$$

and

$$\det(J(E_1)) = \mu(\delta+\mu)(\Omega+\mu)(\mu+\alpha+\gamma+\eta)\left[1-\frac{\Omega\beta\Lambda}{\mu(\Omega+\mu)(\mu+\alpha+\gamma+\eta)}\right]$$

Or $\det(J(E_1)) = \mu(\delta+\mu)(\Omega+\mu)(\mu+\alpha+\gamma+\eta)[1-R_0]>0$ if $R_0<1$.

That is $\det(J(E_1))>0$ if $R_0<1$.

Thus, $trace(J(E_1)<0$ and $\det(J(E_1))>0$.

Hence , by Routh-Hurwitz criteria the theorem is proved.

(B) Global stability of disease-free equilibrium

Theorem 2 . The DFE is globally asymptotically stable (GAS) if $R_0<1$.

Proof . We employ Castillo-Chavez method to prove the theorem . Take equation (1) into two compartments , uninfected and infected individuals, given by $a_1 : \frac{dW}{dt} = F(W,T)$ and $a_2 : \frac{dT}{dt} = G(W,T), G(W,0) = 0$, where , $W=(S,R)\in R_+^2$, $T=(E,I)\in R_+^2$.

Let $$E_1 = (N_0,0), N_0 = (\frac{\Lambda}{\mu}). \qquad (4)$$

Then $E_1=(N_0,0)$ is globally asymptotically stable equilibrium of (4) if the following conditions are satisfied :

b_1: E_1 is globally asymptotically stable for $\frac{dW}{dt} = F(W,0)$, $b_2 : \hat{G}(W,T)\geq 0, (W,T)\in D$,

Where $G(W,T) = AT - \hat{G}(W,T)$, A is a Metzier matrix.

Then we can write $A=M-U$ and D is given by (4). Then

$$\frac{dW}{dt} = F(W,T) = \begin{bmatrix}\Lambda+\delta R+\eta I-\mu S-\beta SI\\ \gamma I-(\delta+\mu)R\end{bmatrix} \text{ and } b_1 : \frac{dW}{dt} = F(W,0) = \begin{bmatrix}\Lambda-\mu S\\ 0\end{bmatrix}$$

and $$\frac{dT}{dt} = G(W,T) = \begin{bmatrix}\beta SI-(\mu+\Omega)E\\ \Omega E-(\mu+\alpha+\gamma+\eta)I\end{bmatrix}, G(W,0)=0.$$

Hence, b_1 is satisfied.

Now for b_2, $A = M-U = \begin{bmatrix}-(\Omega+\mu) & \beta S\\ \Omega & -(\alpha+\mu+\gamma+\eta)\end{bmatrix}$, where

$$M = \begin{bmatrix}0 & \beta\\ 0 & 0\end{bmatrix}, \quad U = \begin{bmatrix}(\mu+\Omega) & 0\\ -\Omega & (\mu+\gamma+\eta+\alpha)\end{bmatrix}.$$

and

$$AT = \begin{bmatrix}-(\Omega+\mu) & \beta S\\ \Omega & -(\alpha+\mu+\gamma+\eta)\end{bmatrix}\begin{bmatrix}E\\ I\end{bmatrix} = \begin{bmatrix}-(\Omega+\mu)E+\beta SI\\ \Omega E-(\alpha+\mu+\gamma+\eta)I\end{bmatrix}$$

Thus, $\hat{G}(W,T) = AT - G(W,T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{G}(W,T)_1 \\ \hat{G}(W,T)_2 \end{bmatrix}$.Hence, $b_2 : \hat{G}(W,T) \geq 0, (W,T) \in D$.

Hence the theorem is proved.

(C) Local stability of endemic equilibrium

Theorem 3. If $R_0 > 1$, then EE is LAS .

Proof. Consider the matrix at endemic equilibrium will be

$$J(E_2) = \begin{bmatrix} -\beta I^* - \mu & 0 & -\beta S^* & \delta \\ \beta I^* & -(\Omega + \mu) & \beta S^* & 0 \\ 0 & \Omega & -(\mu + \alpha + \gamma + \eta) & 0 \\ 0 & 0 & \gamma & -(\delta + \mu) \end{bmatrix}.$$

Now,

$$trace(J(E_2)) = -\mu - \beta I^* - (\Omega + \mu) - (\mu + \alpha + \gamma + \eta) - (\delta + \mu)$$
$$= -(4\mu + \beta I^* + \Omega + \alpha + \gamma + \eta + \delta) < 0$$

and

$$\det(J(E_2)) = (\beta I^* + \mu)(\delta + \mu)\left[(\Omega + \mu)(\mu + \alpha + \gamma + \eta) - \Omega\beta S^*\right] + \beta S^*(\delta + \mu)\Omega\beta I^* - \delta\beta I^*\Omega\gamma > 0 \text{ if } R_0 > 1$$

That is $\det(J(E_2)) > 0$ if $R_0 > 1$.

Thus, $trace(J(E_2)) < 0, \det(J(E_2)) > 0$.

Hence, by Routh-Hurwitz criteria the theorem is proved.

(D) Global stability of endemic equilibrium

Theorem 4. The EE is GAS if $R_0 > 1$.

Proof . Consider the Lyapunov function

$$V(S^*, E^*, I^*, R^*) = (S - S^* - S^* \ln(\frac{S}{S^*})) + (E - E^* - E^* \ln(\frac{E}{E^*})) + (I - I^* - I^* \ln(\frac{I}{I^*})) + (R - R^* - R^* \ln(\frac{R}{R^*}))$$

then,

$$\frac{dV}{dt} = \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{E^*}{E}\right)\frac{dE}{dt} + \left(1 - \frac{I^*}{I}\right)\frac{dI}{dt} + \left(1 - \frac{R^*}{R}\right)\frac{dR}{dt}.$$

Using equations of system (1), then we obtain

$$\frac{dV}{dt} = \left(1 - \frac{S^*}{S}\right)(\Lambda + \delta R + \eta I - \mu S - \beta SI) + \left(1 - \frac{E^*}{E}\right)(\beta SI - (\mu + \Omega)E) + \left(1 - \frac{I^*}{I}\right)(\Omega E - (\mu + \alpha + \gamma + \eta)I)$$
$$+ \left(1 - \frac{R^*}{R}\right)(\gamma I - (\delta + \mu)R)$$

$$\frac{dV}{dt} = \left[\Lambda + \delta R + \mu S^* + \beta S^* I + \beta SI + (\mu + \Omega)E^* + \Omega E + (\mu + \alpha + \gamma + \eta)I^* + \eta I + \gamma I + (\delta + \mu)R^*\right]$$
$$- \left[\mu S + \beta SI + \Lambda\frac{S^*}{S} + \delta R\frac{S^*}{S} + + \eta I\frac{S^*}{S} + (\mu + \Omega)E + \beta SI\frac{E^*}{E} + (\mu + \alpha + \gamma + \eta)I + \Omega E\frac{I^*}{I} + (\delta + \mu)R + \gamma I\frac{R^*}{R}\right]$$

or, $\frac{dV}{dt} = A - B \leq 0$ if $A < B$

where , $A = \left[\Lambda + \delta R + \mu S^* + \beta S^* I + \beta SI + (\mu + \Omega) E^* + \Omega E + (\mu + \alpha + \gamma + \eta) I^* + \eta I + \gamma I + (\delta + \mu) R^* \right]$

and $B = \left[\mu S + \beta SI + \Lambda \frac{S^*}{S} + \delta R \frac{S^*}{S} + \eta I \frac{S^*}{S} + (\mu + \Omega) E + \beta SI \frac{E^*}{E} + (\mu + \alpha + \gamma + \eta) I + \Omega E \frac{I^*}{I} + (\delta + \mu) R + \gamma I \frac{R^*}{R} \right]$

So, $\frac{dV}{dt} \leq 0$ if $A < B$ and $\frac{dV}{dt} = 0$ only if $S = S^*, E = E^*, I = I^*, R = R^*$.

Hence,by Lasalle's invariance principle the theorem is proved .

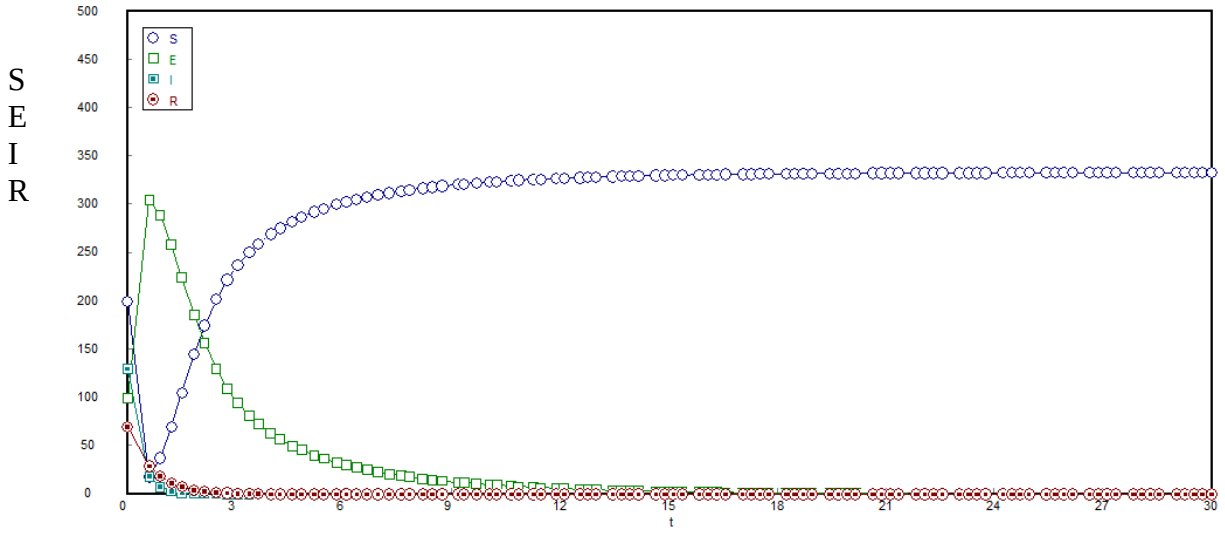
5. Numerical Simulation

Case I : Disease dies out at $R_0 < 1$

$S(0) = 200$, $E(0) = 100$, $I(0) = 130$, $R(0) = 70$, $\Lambda = 500 / year$, $\delta = 0.0125 / year$,

$\eta = 0.2 / year$, $\beta = 1.25 / year$, $\mu = 1.5 / year$, $\Omega = 0.00925 / year$, $\alpha = 1.503 / year$,

$\gamma = 0.0625 / year$, $R_0 = 0.78 < 1$

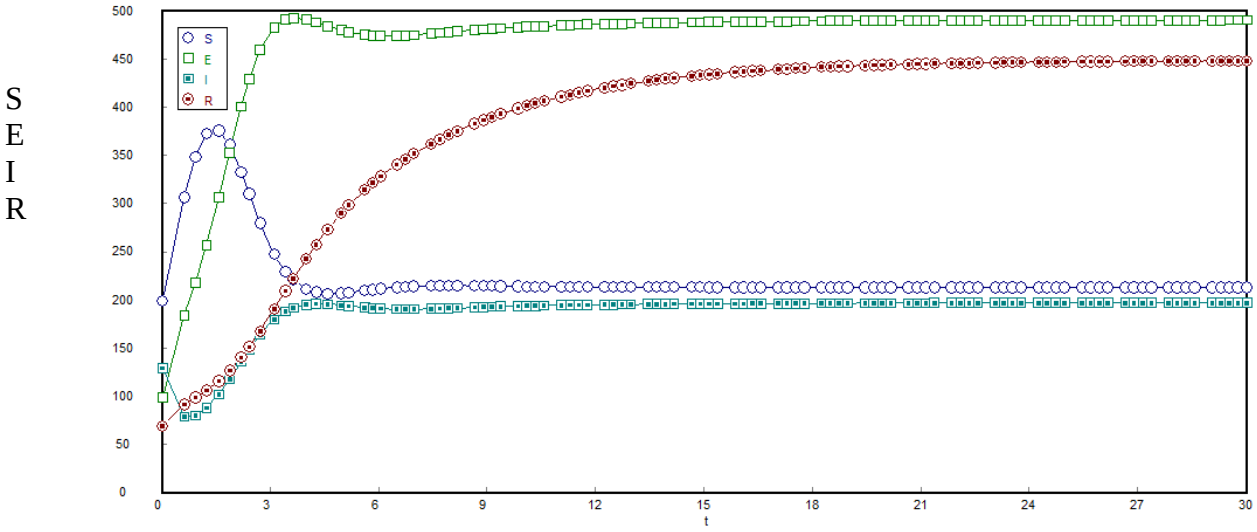


Time t(years)

Figure 2. SEIS Typhoid model when $R_0 < 1$

Case II : Disease persists at $R_0 > 1$

$S(0) = 200$, $E(0) = 100$, $I(0) = 130$, $R(0) = 70$, $\Lambda = 500 / year$, $\delta = 0.125 / year$,
$\eta = 0.02 / year$, $\beta = 0.0125 / year$, $\mu = 0.15 / year$, $\Omega = 0.925 / year$, $\alpha = 1.503 / year$,
$\gamma = 0.625 / year$, $R_0 = 15.6 > 1$



Time t(years)

Figure 3. SEIS Typhoid model when $R_0 > 1$

6. Conclusion

In the present paper, we have formulated SEIS typhoid epidemic model and observed that by case studies how some infected population move towards susceptibility even if some infected individuals goes to recovered class . We have determined equilibrium points(disease-free and endemic) for the model and analyzed its stability criteria . We have studied that by Routh Hurwitz criteria ,both euilibria are locally asymptotically stable if $R_0 < 1$ and $R_0 > 1$ respectively. Globally asymptotic behavior for the one equilibrium point has been discusssed by Castillo-Chavez method and for another equilibrium globalism discussed by Lyapunov-Lasalle's principle for $R_0 < 1$ and $R_0 > 1$ respectively . Finally , we observed that numerically the model has been more strong with graphical representation to decide that when the disease dies out and when the disease perists.

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