ENUMERATION OF RELATIONS AND FUNCTIONS

Dr. R. Sivaraman

Associate Professor, Department of Mathematics, D. G. Vaishnav College, Chennai, India

National Awardee for Popularizing Mathematics among masses

Abstract

Enumeration of Relations and Functions between two finite sets has always been an interesting study in Combinatorics, a branch of mathematics which is seen as art of counting without actual counting. In this paper, we discuss the enumeration of various relations and functions. Interestingly enough, the answers turns out be part of interesting class of numbers.

Keywords*: Equivalence Relations, One-One Functions, Onto Functions, Stirling's Numbers of Second Kind, Bell Numbers, Multinomial coefficients.*

1. Introduction

The concept of relations and functions has been fundamental aspect in understanding almost all branches of higher mathematics like Group Theory, Vector Spaces, Topological Spaces and much more. The relations and functions that can be defined between two given sets which are usually called domain and co-domain are basic tools for mathematicians for exploring higher structures in mathematics. By considering finite domain and co-domain, we will try to enumerate the number of relations and functions that can possibly exist between such sets. This paper focus on such enumeration process. The results obtained were surprisingly related to various class of interesting numbers.

2. Definitions

2.1 A relation (or binary relation) *R* that can be defined between two non-empty sets *A* and *B* is the set of all ordered pairs that can exist between the elements of *A* and *B*.

That is, $R = \{(x, y) / x \in A, y \in B\}$ (2.1)

2.2 A relation between two non-empty sets as defined in (2.1) is called an Equivalence Relation, if it is Reflexive, Symmetric and Transitive (2.2)

It is well known that any equivalence relation induces a partition of the given set. A partition of a set is splitting the whole set in to disjoint subsets whose union is the set itself.

2.3 A function *f* between two sets *A* and *B* is a relation in which every element of *A* has a unique element as image under *f* in *B*. The sets *A* and *B* are called domain and co-domain of *f* respectively. The function between *A* and *B* is denoted by $f : A \to B$. Thus a relation is a function in which every element in the domain possess a unique image in the co-domain.

2.4 Let $f : A \to B$ be a function. The set of all images of *f* is defined as Range of the function *f*. The range of the function *f* is denoted by $f(A)$. From the definition of a function, it is clear that $f(A) \subseteq B$. If $f(A) = B$ then we call *f* as **onto** or **surjective** function. Thus for an **onto** function, every element in the co-domain will possess at least one pre-image in the domain of *f*. If *f* is not onto, then it is called **into** function.

2.5 Let $f : A \to B$ be a function. If the pre-images of every element in the range of *f* are also unique, then *f* is said to be **one-one** function or **injective** function.

2.6 A function $f : A \to B$ which is **one-one** and **onto** is called a **bijection** or **one-one correspondence**. In this case, we say that *A* and *B* are equivalent sets.

We always consider the domain *A* and co-domain *B* to be finite sets for further discussion of this paper. In particular, we assume that *A* contains *m* elements and *B* contains *n* elements. That is, $|A| = m, |B| = n$.

3. In this section we prove two theorems regarding enumeration of relations.

3.1 Theorem 1

The total number of relations that can exist between *A* and *B* such that $|A| = m, |B| = n$ is 2^{mn} (3.1)

Proof: If no element in *A* has any image then we get what is called a null relation which possesses no ordered pair. If all the elements of *A* are mapped on to all the elements of *B* then we get a relation which represent the Cartesian product between *A* and *B*. Clearly, this must be the relation with maximum number of ordered pairs that can exist between the elements of *A* and *B*. We also note that the number of elements in the Cartesian product is $|A \times B| = |A| \times |B| = mn$.

Thus any relation that can be defined between *A* and *B* would be a subset of the Cartesian product. Thus, the total number of relations that can exist between A and B would be the cardinality of the power set of the Cartesian product which contains mn elements. Therefore, using the cardinality of the power set, the number of relations is clearly 2^{mn}. This completes the proof.

3.2 Stirling's Numbers of Second Kind

Let *S* be a set with *n* elements. We define the Stirling's numbers of second kind denoted by

$S(n,k)$ or $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ as the number of partitions of *S* containing exactly *k* parts. That is the Stirling's numbers of second kind represent the number of partitions of a set with *n* elements using *k* non-empty disjoint subsets.

In this sense, it follows that $0 \le k \le n$. In particular, if *n* = 0, *k* = 0 we consider $S(0,0) = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1$. Similarly there is only one possible partition namely the whole set *S* itself if *k* = *n*. Thus, $S(n,n) = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1$. Also, if $k > n$ then there is no possibility of obtaining any partition of *S* with more than *n* non-empty subsets (since the minimum cardinality must be 1). Hence, $S(n,k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0$ if $k > n$.

With the aid of this definition, we can construct a triangle portraying Stirling's numbers of second kind as shown in Figure 1.

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3025	7770	6951	2646	462	36	1	
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

Figure 1: Stirling Numbers of Second Kind Triangle

3.3 Bell Numbers

The sum of each row numbers in Figure 1 containing Stirling numbers of second kind are called Bell's Numbers. If we do so, then from Figure 1, we get 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, . . . We denote the *n*th Bell number by B_n. Thus $B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 15, B_4 = 52, B_5 = 203, B_6 = 877, \ldots$

The sequence of first twenty Bell numbers are given by 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804, 682076806159, 5832742205057, . . .

To know more about Stirling's numbers of second kind and Bell Numbers, see **[1], [2].** The following theorem indicates the need of Bell numbers.

3.4 Theorem 2

The total number of equivalence relations that can be defined on a set with *n* elements is the *n*th Bell number B_n.

Proof: By definition of Stirling numbers of second kind, we know that $S(n,k)$ represent the number of partitions of a set with *n* elements using *k* non-empty disjoint subsets where $0 \le k \le n$. Thus the total number of possible partitions that can be obtained for a set with *n* elements will be sum of all Stirling's numbers of second kind for each value of *k* given by $\sum_{k=0}^{n} S(n,k) \quad (3.2)$. At the same time, $\sum_{k=0}^{n} S(n,k)$ also represents the sum of all numbers in row *n* of the Stirling numbers of second kind triangle of Figure 1. But by definition of Bell numbers, this sum is precisely the *n*th Bell number by B_n. Hence, $B_n = \sum_{k=0}^{n} S(n,k) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \quad (3.3)$.

We know that any equivalence relation defined on a set produces a partition of that set. Hence, the total number of equivalence relations defined on a set with *n* elements must be same as that of the number of partitions that can be obtained for that set. But the total number of partitions for a set with *n* elements from (3.3) is the *n*th Bell number by B_n. Thus, the total number of equivalence relations on a set with *n* elements is precisely B_n. This completes the proof.

4. In this section, we discuss theorems concerning enumeration of various kinds of functions.

4.1 Theorem 3

If $f : A \to B$ is a onto (surjective) function where $|A| = m, |B| = n$ then $m \ge n$ (4.1)

Proof: According to the definition of onto function in section 2.4, we see that *f* is onto if every element in the co-domain *B* should have at least one pre-image in the domain *A*. Since $|A| = m, |B| = n$, we must have $m \ge n$.

4.2 Theorem 4

If $f : A \to B$ is a one-one (injective) function where $|A| = m, |B| = n$ then $m \le n$ (4.2)

Proof: If *f* is one-one, then by definition in section 2.5, the pre-image of every element in the range of *f* must have unique pre-image in the domain *A*. Thus, if the range of *f* is $f(A)$ then each of the *p* elements in the domain *A* will be mapped under *f* to some *p* elements in the range $f(A)$. Hence, $|f(A)| = m$. Since $f(A) \subseteq B$ it follows that, $|f(A)| \le |B|$ giving $m \le n$. This completes the proof.

4.3 Theorem 5

If $f : A \to B$ where $|A| = m, |B| = n$ is a bijection (one-one and onto) then $m = n$ (4.3)

Proof: From (4.1) of theorem 3, we know that if *f* is onto then $m \geq n$. Similarly, from (4.2) of Theorem 4, we know that if *f* is one-one then $m \leq n$. Hence, if *f* is both one-one and onto i.e. if *f* is a bijection then $m \geq n$ as well as $m \leq n$. Thus *m* = *n*. This completes the proof.

5. In this section, we provide methods for number of functions that can exist between two finite sets *A* and *B* such that $|A| = m, |B| = n$.

5.1 Theorem 6

Let $f : A \to B$ where $|A| = m, |B| = n$. Then the number of functions between *A* and *B* is n^m (5.1)

Proof: If $f : A \to B$ is a function, then each element in the domain of *f* should be mapped on to any one of the *n* elements in *B*. Thus each element of *A* has *n* choices of images in *B*. Hence by multiplication rule of counting, the total number of possible functions *f* that can exist between *A* and *B* would be $n \times n \times n \times \cdots \times n \times n \ \ (m\,times) = n^m$. This completes the proof.

5.2 Theorem 7

Let $f : A \to B$ where $|A| = m, |B| = n$. Then the number of one-one functions (injectives) between *A* and *B* is given by

$$^nP_m = \frac{n!}{(n-m)!} \quad (5.2)$$

Proof: We will list the elements of *A* as $A = \{x_1.x_2...., x_m\}$. Since *f* is one-one, by (4.2) of Theorem 4, we have $m \leq n$. Hence $(n-m)!$ is well defined. Since $x_1 \in A$, there are *n* possible choices for $f(x_1) \in B$. Since *f* is one-one, for $x_2 \in A$ there are *n* – 1 possible choices (since x_2 cannot have same image as that of x_1).such that $f(x_2) \in B$. Similarly, for $x_3 \in A$ there are *n* – 2 possible choices such that $f(x_3) \in B$. Proceeding in same fashion since $m \leq n$, for $x_p \in A$ there are $n-(m-1) = n-m+1$ possible choices such that $f(x_p) \in B$. Hence by multiplication theorem of counting, the total number of one-one functions (injectives) that can exist between *A* and *B* is $n \times (n-1) \times (n-2) \times \cdots \times (n-m+1) = {}^nP_m = \frac{n!}{(n-m)!}$. This completes the proof.

5.3 Theorem 8

Let $f : A \to B$ where $|A| = m, |B| = n$. Then the number of onto functions (surjectives) between *A* and *B* is given by

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)^m = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m \quad (5.3)$$

Proof: From (5.1) of Theorem 6, we know that there are totally n^m functions that exist between *A* and *B*. To make sure that we need only onto functions between *A* and *B*, we want to exclude functions among n^m which are not onto.

First, we count the number of functions which maps only to *n* – 1 elements of *B*. Such functions will clearly be not onto as we have left one element from *B*. The number of functions which maps to only (*n* – 1) elements of *B* by (5.1) would be $(n-1)^m$. There are $\binom{n}{1}$ choices to leave 1 element from *n* elements in *B*. Hence the number of functions which are not onto but mapping on to any of (*n* – 1) elements of *B*, by multiplicative rule of counting would be $\binom{n}{1}(n-1)^m$. Similarly there are $\binom{n}{2}$ choices of leaving 2 elements from *n* elements of *B* and number of functions which are not onto then (by (5.1)) would be $(n-2)^m$. So, the number of functions which are not onto but mapping on to any of (n – 2) elements of *B*, by multiplicative rule of counting would be $\binom{n}{2}(n-2)^m$. Proceeding in similar fashion, we see that at the most we can leave at the most n – 1 elements of *B* in $\binom{n}{n-1}$ ways and number of such functions which are not onto would be $\binom{n}{n-1}(n-(n-1))^m = \binom{n}{n-1}1^m$.

Now the total number of onto functions that exist between *A* and *B* can be obtained by using Principle of Inclusion and Exclusion (PIE). Thus the total number of onto functions is given by

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m = \sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)^m$$

This completes the proof.

We end this paper with the following alternative formula for counting onto functions using Stirling's numbers of second kind. These numbers are displayed in Figure 1. Here is an recursive formula to determine Stirling's numbers of second kind $S(m,n)$ where $m \geq n$.

If $m \geq n \geq 2$, we have $S(m+1,n) = S(m,n-1) + nS(m,n)$ (5.4)

We now present a theorem for counting number of onto functions using the Stirling's numbers of second kind. This theorem is exactly the same as Theorem 8 but provided in different perspective.

5.4 Theorem 9

Let $f: A \to B$ where $|A| = m, |B| = n$. Then the number of onto functions (surjectives) between *A* and *B* is $n!S(m,n)$ (5.5) where $S(m,n)$ are Stirling's numbers of second kind.

Proof: We first treat the elements of the domain *A* as labeled balls and elements of the co-domian *B* as labeled urns. Then, the number of onto functions between A and B can be viewed as the number of possible ways that *m* labeled balls can be distributed among *n* labeled urns such that no urn is non-empty.

We note that distributing *m* labeled balls among *n* urns is the number of partitions of *A* containing *m* elements in to exactly *n* parts. We also note that for onto functions we should have (by (4.1) of theorem 3), $m \geq n$. Hence the distribution of *m* labeled balls among *n* urns represents the Stirling's number of second kind of the form $S(m,n)$.

Once the balls have been distributed, there are $n!$ ways to label the urns. Hence by multiplication rule of counting, the total number of onto functions that exist between *A* and *B* must be $n!S(m,n)$. This completes the proof.

Finally, we note that $n!S(m,n) = \sum\binom{m}{r_1\, r_2\, r_3 \cdots r_n} = \sum\frac{m!}{r_1! \times r_2! \times r_3! \times \cdots \times r_n!}$ (5.5) where $r_1 + r_2 + r_3 + \cdots + r_n = n$,

Hence, the number of onto functions by Theorem 9 would be $\sum\binom{m}{r_1\, r_2\, r_3 \cdots r_n}$

Thus, we see the number of onto functions can also be represented by sum of multinomial coefficients. As an illustration the number of onto functions that exist between *A* and *B* where $|A| = 5, |B| = 3$ according to (5.5) would be either $3!S(5,3)$ or $\sum\binom{5}{r_1\, r_2\, r_3}$ where $r_1 + r_2 + r_3 = 5$.

Now using Table 1, we find that $S(5,3) = 25$. We can also use (5.4) to get this. Hence $3!S(5,3) = 6 \times 25 = 150$. Now we will get the same result using multinomial coefficient formula.

$$\sum \binom{5}{r_1\, r_2\, r_3} = \binom{5}{1\,2\,2} + \binom{5}{2\,1\,2} + \binom{5}{2\,2\,1} + \binom{5}{1\,1\,3} + \binom{5}{1\,3\,1} + \binom{5}{3\,1\,1} = (3 \times 30) + (3 \times 20) = 150.$$

So among $3^5 = 243$ possible functions that exist between *A* and *B*, 150 functions would be onto.

6. Conclusion

The enumeration of number of relations, equivalence relations, functions, one-one functions, onto functions, bijections had been done through various theorems in this paper. Interestingly enough, we saw that the number of equivalence relations that can be defined on a set with *n* elements is the *n*th Bell number. The number of one-one functions depends on the number of ways of permuting *m* elements among *n* elements. Theorems 8 and 9 provide two equivalent ways of counting number of onto functions between *A* and *B*. We witnessed that the number of onto functions depends on Factorials and Stirling's numbers of second kind as well as multinomial coefficients. Connection between completely unrelated concepts is the beauty of mathematics and this aspect is repeatedly reflected in this paper. Moreover, this paper provides the complete enumeration of relations and functions that can exist between two finite sets.

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