Some Inequalities for m-convex functions VIA Fractional Integrals

Tehmina Afzal[*], Liaqat Ali[**], Muhammad Aslam[***], Yaqoub Ahmed Khan[****]

[*] Department of Mathematics, Govt. College University Lahore, Pakistan
[**] Department of Mathematics, Govt. MAO College Lahore, Pakistan
[***] Department of Mathematics, Govt. College University Lahore, Pakistan
[****] Department of Mathematics, GIC Lahore, Pakistan

Abstract- Fractional integrals has gained a very important place in mathematical analysis. Main aim of this research article is to derive some results for m-convex functions related to Hermite Hadamard and fejer type inequalities using the Katugampola fractional integrals

Index Terms- Riemann-Liouville Fractional Integrals, Katugampola Fractional Integrals, Convex Functions, m-convex Functions

I. Introduction

The notion of inequalities is one of the important aspects of mathematics having a wide range of applications in the other areas of mathematics and other sciences as well. Several mathematicians have been working on the notion of inequalities with different types of convex functions satisfying certain integral conditions, for ready reference one can see [11, 12, 10, 9, 16, 26, 1, 2]. In this regard, HermiteHadamard type inequalities are very well know which have been studied, refined and generalized for different types of convex functions under different circumstances and parameters. To study A considerable number of integral inequalities of the Hermite – Hadamard type for convex functions via fractional integrals have been established (see [6, 5, 23, 8, 12, 10, 9, 16, 1, 2]. S.S Dragomir and G.H Toader [7, 24] introduced the concept of $m-$convex functions.

We now give a brief introduction of the basic concepts and terminology which will be very useful for the sequel.

Let $I=[a,b]$ be a closed interval, a function $f:I\to R$ is said to be convex, if

$$f(\alpha x+(1-\alpha)y)\leq \alpha f(x)+(1-\alpha)f(y),$$

for all $x,y\in[a,b],\ \alpha\in[0,1]$. For more on convex functions and related aspects in detail one can see [25, 15].

A function $f:[0,b]\to R$ is called m-convex $0\leq m\leq 1$, if $\forall x,y\in[0,b], t\in[0,1]$, we have

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

when $m=1$ the concept of convex functions can be recaptured. For more on m-convexity see [7, 20].

Let $f:[a,b]\to R$ be a convex function $a,b\in I$, $a<b$. Then the inequality

$$f(\frac{a+b}{2})\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}$$

is known as Hermite-Hadamard inequality [6]. Fejer gave an idea of the generalization of the Hermite-Hadamard inequality. Let a $f:[a,b]\to R$ be a convex and $g:[a,b]\to R$ a positive and integrable function which is also symmetric about the $\frac{a+b}{2}$. Then the inequality

$$f(\frac{a+b}{2})\int_a^b g(x)dx\leq\int_a^b f(x)g(x)dx\leq\frac{f(a)+f(b)}{2}\int_a^b g(x)dx$$

is known as Hermite-Hadamard-Fejer inquality. If we take $g(x)=1$ in Hermite-Hadamard-Fejer inequality, we get the Hermite-Hadamard inequality. Consider $g\in[a,b]$. Then Riemann-Liouville fractional integrals [5] having order $\alpha>0$ with $a..0$ are defined as

$$J_{a+}^{\alpha}g(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}g(t)dt, x>a$$

$$J_{b-}^{\alpha}g(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}g(t)dt, x<b$$

where Γ is the Gamma function.

To see more on the generalization of Hermite-Hadamard type inequalities we refer [17, 27, 14, 21, 22, 19, 8]. Next we state some results with references which will be frequently discussed and recalled in the sequel.

Theorem 1.1 [23]

Let $f:[a,b]\to R$ be a positive function with $0\leq a<b$ and $f\in L[a,b]$ If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold

$$f(\frac{a+b}{2})\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)]\leq\frac{f(a)+f(b)}{2}$$ where $\alpha>0$.

Definition 1.2 [18]

Let $[a,b]\subseteq R$ be a finite interval. Then the left-side and right-side Katugampola fractional integrals having order $\alpha>0$ with $a<x<b$ and $p>0$, are defined as follows

$${}^{p}I_{a+}^{\alpha}f(x)=\frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_a^x t^{p-1}(x^p-t^p)^{\alpha-1}f(t)dt$$

$${}^{p}I_{b-}^{\alpha}f(x)=\frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_x^b t^{p-1}(t^p-x^p)^{\alpha-1}f(t)dt$$

Definition 1.3 [17]

Let f be a function of order α, where $\alpha>0,\ n-1\leq\alpha\leq n,$ $n\in N$ with $a<x<b$. The left-side and right-side Hadamard fractional integrals are defined as

$$H_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (\ln\frac{x}{t})^{\alpha-1}\frac{f(t)}{t}dt$$

$$H_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\ln\frac{x}{t})^{\alpha-1}\frac{f(t)}{t}dt$$

Theorem 1.4[17] If $\alpha > 0$ and $p > 0$. Then for $x > a$, we have

$$\lim_{p\to 0^+}({}^pI_{a+}^{\alpha} f(x)) = H_{a+}^{\alpha} f(x)$$

$$\lim_{p\to 1}({}^pI_{a+}^{\alpha} f(x)) = J_{a+}^{\alpha} f(x)$$

Lemma 1.5[4]

Let $\alpha > 0$, $\rho > 0$ and $f:[a^p,b^p]\to R$ be a differentiable mapping on (a^p,b^p) where $0 \le a < b$. Then the following equality holds if the fractional integrals exist.

$$\frac{f(a^p)+f(b^p)}{\alpha p} - \frac{p^{\alpha-1}.\Gamma(\alpha)}{(b^p-a^p)^{\alpha}}[{}^pI_{a+}^{\alpha} f(b^p) + {}^pI_{b-}^{\alpha} f(a^p)]$$

$$= \frac{(b^p-a^p)}{\alpha}\int_0^1 t^{p(\alpha+1)-1}[f'(t^pb^p + a^p(1-t^p)) - f'(t^pa^p+(1-t^p)b^p)]dt$$

Theorem 1.6[4]

Let $f:[a^p,b^p]\to R$ be a differentiable mapping with $0 \le a < b$. If f' is differentiable on (a^p,b^p). If f is also a convex function, Then following inequality holds.

$$\left|\frac{f(a^p)+f(b^p)}{2} - \frac{p^{\alpha}.\Gamma(\alpha+1)}{2(b^p-a^p)^{\alpha}}[{}^pI_{a+}^{\alpha} f(b^p) + {}^pI_{b-}^{\alpha} f(a^p)]\right|$$

$$\le \frac{(b^p-a^p)^2}{2(\alpha+2)(\alpha+1)}[\alpha+\frac{1}{2^{\alpha}}]\sup_{\xi\in[a^p,b^p]}|f''(\xi)|]$$

Theorem 1.7[4]

Let $f:[a^p,b^p]\to R$ be a differentiable mapping on (a^p,b^p) with $0 \le a < b$. If $|f'|$ is convex on $[a^p,b^p]$, then following inequality holds.

$$\left|\frac{f(a^p)+f(b^p)}{2} - \frac{p^{\alpha}.\Gamma(\alpha+1)}{2(b^p-a^p)^{\alpha}}[{}^pI_{a+}^{\alpha} f(b^p) + {}^pI_{b-}^{\alpha} f(a^p)]\right|$$

$$\le \frac{(b^p-a^p)}{2(\alpha+1)}(|f'(b^p)|+|f'(a^p)|)$$

Lemma 1.8[4]

Let $f:[a^p,b^p]\to$ R be a differentiable mapping on (a^p,b^p) with $0 \le a < b$ .Then the following inequality holds

$$\frac{f(a^p)+f(b^p)}{2} - \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p-a^p)^{\alpha}}[{}^pI_{b-}^{\alpha} f(a^p) + {}^pI_{a+}^{\alpha} f(b^p)]$$

$$= p(\frac{b^p-a^p}{2})\int_0^1[(1-t^p)^{\alpha} - t^{p\alpha}]t^{p-1}f'(t^pa^p+(1-t^p)b^p)dt$$

Theorem 1.9[4]

Let $f:[a^p,b^p]\to R$ be a differentiable mapping with $0 \le a < b$. If $|f'|$ is differentiable on (a^p,b^p). If f' is convex on $[a^p,b^p]$ Then following inequality holds

$$\left|\frac{f(a^p)+f(b^p)}{2} - \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p-a^p)^{\alpha}}[{}^pI_{b-}^{\alpha} f(a^p) + {}^pI_{a+}^{\alpha} f(b^p)]\right|$$

$$\le \frac{b^p-a^p}{2(\alpha+1)}[|f'(a^p)|+|f'(b^p)|](1-\frac{1}{2^{\alpha}})$$

It is important to note some typo mistakes were found in the results of referred paper [4], therefore following remarks are being included so that reader may not face any confusion at any stage in the sequel.

Remarks

The above recalled results Lemma 1.5, Theorem 1.6, Theorem 1.7, Lemma 1.8, and Theorem 1.9 are the corrected versions of equation number 14, Theorem 2.2, Theorem 2.3, Lemma 2.4 and Theorem 2.5 of [4] respectively.

II. MAIN RESULTS

Theorem 2.1 Let $\alpha > 0$ and $p > 0$. Let $f:[a^p, m^pb^p]\to R$ be a positive function with $0 \le a < mb$. If f is also an m-convex function on $[a, mb]$, then the following inequality holds

$$f(\frac{a^p+m^pb^p}{2}) \le \frac{p^{\alpha}.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^{\alpha}}[{}^pI_{a+}^{\alpha} f(m^pb^p) + (m^p)^{\alpha+1}{}^pI_{b-}^{\alpha} f(\frac{a^p}{m^p})]$$

$$\le \frac{\alpha}{2(\alpha+1)}[\{f(a^p)-(m^p)^2.f(\frac{a^p}{(m^p)^2})\} + \frac{m^p}{2}\{f(b^p)+(m^p).f(\frac{a^p}{(m^p)^2})\}] \quad (1)$$

Proof. As f is m-convex function, for $t\in[0,1]$ and $x, y \in [a, mb]$, we have

$$f(\frac{x^p+m^py^p}{2}) \le \frac{f(x^p)+m^pf(y^p)}{2}$$

Setting $x^p = t^pa^p + m^p(1-t^p)b^p$; $y^p = t^pb^p + \frac{1}{m^p}(1-t^p)a^p$

$$2f(\frac{a^p+m^pb^p}{2}) \le [f(t^pa^p+m^p(1-t^p)b^p) + m^pf(t^pb^p+\frac{(1-t^p)a^p}{m^p})] \quad (2)$$

Multiplying both sides (2) by $t^{\alpha p-1}$ and integrating from 0 to 1 with respect to t, we get

$$2f(\frac{a^p+m^pb^p}{2})\int_0^1 t^{\alpha p-1}dt \le \int_0^1 t^{\alpha p-1}f(t^pa^p+m^p(1-t^p)b^p)dt + m^p\int_0^1 t^{\alpha p-1}f(t^pb^p+a^p\frac{(1-t^p)}{m^p})dt$$

and therefore

$$\frac{2}{\alpha p}f(\frac{a^p+m^pb^p}{2}) \le \int_0^1 t^{\alpha p-1}f(t^pa^p+m^p(1-t^p)b^p)dt + m^p\int_0^1 t^{\alpha p-1}f(t^pb^p+\frac{a^p(1-t^p)}{m^p})dt \quad (3)$$

We can also write

$$\frac{2}{\alpha p}f(\frac{a^p+m^pb^p}{2}) \le \int_a^{mb}\frac{(m^pb^p-x^p)^{\alpha-1}}{(m^pb^p-a^p)^{\alpha}}x^{p-1}f(x^p)dx + m^p\int_{\frac{a}{m}}^{b}[\frac{y^p-\frac{a^p}{m^p}}{b^p-\frac{a^p}{m^p}}]^{\alpha}.f(y^p).\frac{y^{p-1}}{(y^p-\frac{a^p}{m^p})}.dy \quad (4)$$

$$\frac{2}{\alpha p} f(\frac{a^p + m^p b^p}{2}) \le \frac{p^{\alpha-1}.\Gamma(\alpha)}{2(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1p}\, I^\alpha_{b-} f(\frac{a^p}{m^p})] \quad (5)$$

$$f(\frac{a^p + m^p b^p}{2}) \le \frac{p^\alpha.\Gamma(\alpha+1)}{2(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1p} I^\alpha_{b-} f(\frac{a^p}{m^p})]$$

This established 1st inequality. Next, we prove 2nd inequality. If f is m-convex then for $t \in [0,1]$

$$f(t^p a^p + m^p(1-t^p)b^p) \le t^p f(a^p) + m^p(1-t^p) f(b^p) \quad (6)$$

and from (6), we have inequality

$$m^p f(t^p b^p + \frac{(1-t^p)a^p}{.m^p}) \le m^p t^p f(b^p) + (m^p)^2(1-t^p) f(\frac{a^p}{(m^p)^2}) \quad (7)$$

Adding (6) and (7) inequalities,we have

$$f(t^p a^p + m^p(1-t^p)b^p) + m^p f(t^p b^p + \frac{(1-t^p)a^p}{m^p}) \le t^p f(a^p) + m^p f(b^p) + (m^p)^2(1-t^p).f(\frac{a^p}{(m^p)^2}) \quad (8)$$

Multiplying both sides of (8) by $\frac{\alpha}{2} t^{\alpha p-1}$ and integrating with respect to t from 0 to 1

$$\frac{p^\alpha \Gamma(\alpha+1)}{2(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(b^p m^p) + (m^p)^{\alpha+1}.{}^p I^\alpha_{b-} f(\frac{a^p}{m^p})] \le \frac{\alpha}{2(\alpha+1)}[f(a^p) - (m^p)^2 f(\frac{a^p}{(m^p)^2})] + \frac{m^p}{2}[f(b^p) + (m^p).f(\frac{a^p}{(m^p)^2})]$$

which proves the required result.

Corollary 2.2 Taking $m=1$ in Theorem 2.1 it becomes the result of [4].

Remark 2.3 If we take $m=1$ along with $\alpha=1$ in Theorem 2.1 then we obtain an inequality which is established in [23].

Theorem 2.4 Let $f:[a^p, m^p b^p] \to R$ be a differentiable mapping with $0 \le a < mb$. If $f^{'}$ is differentiable on $(a^p, m^p b^p)$, then following inequality holds.

$$|\frac{f(a^p) + m^p f(b^p)}{2} - \frac{p^\alpha.\Gamma(\alpha+1)}{2(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1p} I^\alpha_{b-} f(\frac{a^p}{m^p})]| \le \frac{(m^p b^p - a^p)}{2m^p} \sup_{\xi\in[a^p, b^p m^p]} |f''(\xi)| [\frac{((m^p b^p - a^p)(m^p+1))}{\alpha+2} + \{\frac{2((m^p)^2 b^p - a^p)^{\alpha+2}}{(m^p b^p - a^p)^{\alpha+1}(m^p+1)^{\alpha+1}(\alpha+1)(\alpha+2)}\} - (\frac{(m^p)^2 b^p - a^p}{\alpha+1})] \quad (9)$$

Proof. Using R.H.S of inequality (3) and (5)

$$\frac{p^{\alpha-1}.\Gamma(\alpha)}{(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1p} I^\alpha_{b-} f(\frac{a^p}{m^p})]$$

$$= \int_0^1 t^{\alpha p-1} f(t^p a^p + m^p(1-t^p)b^p)dt + m^p \int_0^1 t^{\alpha p-1} f(t^p b^p + a^p \frac{(1-t^p)}{m^p})dt \quad (10)$$

Integrating by parts, we obtain

$$\frac{f(a^p) + m^p f(b^p)}{\alpha p} - \frac{p^{\alpha-1}.\Gamma(\alpha)}{(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1}[{}^p I^\alpha_{b-} f(\frac{a^p}{m^p})]] = \frac{(m^p b^p - a^p)}{\alpha}\int_0^1 t^{p(\alpha+1)-1}[f^{'}(t^p b^p + \frac{a^p(1-t^p)}{m^p}) - f^{'}(t^p a^p + m^p(1-t^p)b^p)].dt \quad (11)$$

In the view of Mean Value Theorem, using (11), we get

$$|\frac{f(a^p) + m^p f(b^p)}{\alpha p} - \frac{p^{\alpha-1}.\Gamma(\alpha)}{(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1}[{}^p I^\alpha_{b-} f(\frac{a^p}{m^p})]]| \le \frac{(m^p b^p - a^p)}{\alpha}\int_0^1 t^{p(\alpha+1)-1} |[(\frac{a^p}{m^p} - m^p b^p) + t^p(1+\frac{1}{m^p})(m^p b^p - a^p)]|.|f''\xi(t)|\,dt$$

Take $\beta = t^p = \frac{(m^p)^2 b^p - a^p}{(m^p b^p - a^p)(m^p+1)}$. As $\xi(t) \in (a^p, m^p b^p)$, therefore the right side of the last expression becomes

$$\le \frac{(m^p b^p - a^p)}{\alpha}\sup_{\xi\in[a^p, b^p m^p]}|f''(\xi)|[(m^p b^p - \frac{a^p}{m^p})\int_0^{(\beta)^{1/p}} t^{p(\alpha+1)-1}dt - (1+\frac{1}{m^p})(m^p b^p - a^p)\int_0^{(\beta)^{1/p}} t^{p(\alpha+2)-1}dt + (\frac{a^p}{m^p} - m^p b^p)\int_{\beta^{1/p}}^1 t^{p(\alpha+1)-1}dt + (1+\frac{1}{m^p})(m^p b^p - a^p)\int_{\beta^{1/p}}^1 t^{p(\alpha+2)-1}dt]$$

So we obtain the final result

$$|\frac{f(a^p) + m^p f(b^p)}{2} - \frac{p^\alpha.\Gamma(\alpha+1)}{2(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1}[{}^p I^\alpha_{b-} f(\frac{a^p}{m^p})]]| \le \frac{(m^p b^p - a^p)}{2m^p}\sup_{\xi\in[a^p, b^p m^p]}|f''(\xi)|[\frac{((m^p b^p - a^p)(m^p+1))}{\alpha+2} + (\frac{2((m^p)^2 b^p - a^p)^{\alpha+2}}{(m^p b^p - a^p)^{\alpha+1}(m^p+1)^{\alpha+1}(\alpha+1)(\alpha+2)}) - (\frac{(m^p)^2 b^p - a^p}{\alpha+1})]$$

Corollary 2.5 Taking m = 1 in Theorem 2.4 we obtain Theorem 1.6 [Theorem 2.2, 4]. The result is as follows

$$|\frac{f(a^p) + f(b^p)}{2} - \frac{p^\alpha.\Gamma(\alpha+1)}{2(b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(b^p) + {}^p I^\alpha_{b-} f(a^p)]| \le \frac{(b^p - a^p)^2}{2(\alpha+1)(\alpha+2)}(\alpha + \frac{1}{2^\alpha}) \sup_{\xi\in[a^p, b^p]} |f''(\xi)|$$

Theorem 2.6 Let $f:[a^p, m^p b^p] \to$ R be a differentiable mapping on $(a^p, m^p b^p)$ with $0 \le a < mb$. If $|f'|$ is m-convex on $[a^p, m^p b^p]$, then following inequality holds.

$$|\frac{f(a^p) + m^p f(b^p)}{2} - \frac{p^\alpha.\Gamma(\alpha+1)}{2(m^p b^p - a^p)^\alpha}[{}^p I^\alpha_{a+} f(m^p b^p) + (m^p)^{\alpha+1}[{}^p I^\alpha_{b-} f(\frac{a^p}{m^p})]]|$$

$$\leq \frac{(m^pb^p-a^p)(1-m^p)}{2(\alpha+2)}.[|f'(b^p)|-\frac{|f'(a^p)|}{m^p}]$$

$$+\frac{(m^pb^p-a^p)}{2(\alpha+1)}.[m^p|f'(b^p)|+\frac{|f'(a^p)|}{m^p}]$$

Proof. Using (11)

$$\frac{f(a^p)+m^pf(b^p)}{\alpha p}-\frac{p^{\alpha-1}.\Gamma(\alpha)}{(m^pb^p-a^p)^\alpha}[{}^pI^\alpha_{a+}f(m^pb^p)+(m^p)^{\alpha+1}[{}^pI^\alpha_{b-}f(\frac{a^p}{m^p})]]$$

$$=\frac{(m^pb^p-a^p)}{\alpha}\int_0^1 t^{p(\alpha+1)-1}[f'(t^pb^p+\frac{a^p(1-t^p)}{m^p})-f'(t^pa^p+m^p(1-t^p)b^p)]dt$$

By m-convexity of $|f'|$, we can write

$$|\frac{f(a^p)+m^pf(b^p)}{\alpha p}-\frac{p^{\alpha-1}.\Gamma(\alpha)}{(m^pb^p-a^p)^\alpha}[{}^pI^\alpha_{a+}f(m^pb^p)+(m^p)^{\alpha+1}[{}^pI^\alpha_{b-}f(\frac{a^p}{m^p})]]|$$

$$\leq\frac{(m^pb^p-a^p)}{\alpha}\int_0^1 t^{p(\alpha+1)-1}|f'(t^pb^p+\frac{a^p(1-t^p)}{m^p})-f'(t^pa^p+m^p(1-t^p)b^p)|.dt$$

$$|\frac{f(a^p)+m^pf(b^p)}{\alpha p}-\frac{p^{\alpha-1}.\Gamma(\alpha)}{(m^pb^p-a^p)^\alpha}[{}^pI^\alpha_{a+}f(m^pb^p)+(m^p)^{\alpha+1}[{}^pI^\alpha_{b-}f(\frac{a^p}{m^p})]]|$$

$$\leq\frac{(m^pb^p-a^p)(1-m^p)}{p\alpha(\alpha+2)}.[|f'(b^p)|-\frac{|f'(a^p)|}{m^p}]$$

$$+\frac{(m^pb^p-a^p)}{\alpha p(\alpha+1)}.[m^p|f'(b^p)|+\frac{|f'(a^p)|}{m^p}]$$

Thus, the final result is

$$|\frac{f(a^p)+m^pf(b^p)}{2}-\frac{p^{\alpha}.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^\alpha}[{}^pI^\alpha_{a+}f(m^pb^p)+(m^p)^{\alpha+1}[{}^pI^\alpha_{b-}f(\frac{a^p}{m^p})]]|$$

$$\leq\frac{(m^pb^p-a^p)(1-m^p)}{2(\alpha+2)}.[|f'(b^p)|-\frac{|f'(a^p)|}{m^p}]$$

$$+\frac{(m^pb^p-a^p)}{2(\alpha+1)}.[m^p|f'(b^p)|+\frac{|f'(a^p)|}{m^p}]$$

Corollary 2.7 Taking m=1 in Theorem 2.6, we obtain Theorem 1.7 which is the Theorem 2.3 of [4].

Lemma 2.8 Let $f:[a^p,m^pb^p]\to$ R be a differentiable mapping on (a^p,m^pb^p) with $0\leq a<mb$. Then the following equality holds .

$$\frac{f(a^p)+f(m^pb^p)}{2}-\frac{p^\alpha.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^\alpha}.[{}^pI^\alpha_{mb-}f(a^p)+{}^pI^\alpha_{a+}f(m^pb^p)]$$

$$=p(\frac{m^pb^p-a^p}{2})\int_0^1[(1-t^p)^\alpha-t^{p\alpha}].t^{p-1}.f'(t^pa^p+m^p(1-t^p)b^p)dt$$

Proof. $\int_0^1(1-t^p)^\alpha.t^{p-1}.f'(t^pa^p+m^p(1-t^p)b^p)dt$

Integrating by parts, we get

$$=\frac{(1-t^p)^\alpha f(t^pa^p+m^p(1-t^p)b^p)}{p(a^p-m^pb^p)}|_0^1$$

$$-\frac{\alpha}{(a^p-m^pb^p)}\int_0^1(1-t^p)^{\alpha-1}.t^{p-1}.f(t^pa^p+m^p(1-t^p)b^p)dt$$

$$=\frac{f(m^pb^p)}{p(m^pb^p-a^p)}-\frac{\alpha}{(m^pb^p-a^p)}\int_0^1(1-t^p)^{\alpha-1}.t^{p-1}.f(t^pa^p+m^p(1-t^p)b^pdt$$

$$=\frac{f(m^pb^p)}{p(m^pb^p-a^p)}-\frac{\alpha}{(m^pb^p-a^p)^{\alpha+1}}\int_a^{mb}(x^p-a^p)^{\alpha-1}.f(x^p).x^{p-1}dx$$

$$\int_0^1(1-t^p)^\alpha.t^{p-1}.f'(t^pa^p+m^p(1-t^p)b^p)dt$$

$$=\frac{f(m^pb^p)}{p(m^pb^p-a^p)}-\frac{\alpha p^{\alpha-1}.\Gamma(\alpha)}{(m^pb^p-a^p)^{\alpha+1}}\ {}^pI^\alpha_{mb-}f(a^p) \quad (12)$$

Similarly, we can also see that

$$-\int_0^1 t^{p\alpha}.t^{p-1}.f'(t^pa^p+m^p(1-t^p)b^p)dt$$

$$=\frac{f(a^p)}{p(m^pb^p-a^p)}-\frac{\alpha p^{\alpha-1}.\Gamma(\alpha)}{(m^pb^p-a^p)^{\alpha+1}}.{}^pI^\alpha_{a+}f(m^pb^p) \quad (13)$$

From (12) and (13), we have

$$\frac{f(a^p)+f(m^pb^p)}{2}-\frac{p^\alpha.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^\alpha}.[{}^pI^\alpha_{mb-}f(a^p)+{}^pI^\alpha_{a+}f(m^pb^p)]$$

$$=p(\frac{m^pb^p-a^p}{2})\int_0^1[(1-t^p)^\alpha-t^{p\alpha}].t^{p-1}.f'(t^pa^p+m^p(1-t^p)b^p)dt$$

Corollary 2.9 Consider m=1 in Lemma 2.8, we get Lemma 1.8 which is the Lemma 2.4 of [4].

Theorem 2.10 Let $f:[a^p,m^pb^p]\to R$ be a differentiable mapping with $0\leq a<mb$. If f' is differentiable on (a^p,m^pb^p) .If $|f'|$ is m-convex on $[a^p,m^pb^p]$ Then following inequality holds.

$$|\frac{f(a^p)+f(m^pb^p)}{2}-\frac{p^\alpha.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^\alpha}.[{}^pI^\alpha_{mb-}f(a^p)+{}^pI^\alpha_{a+}f(m^pb^p)]|$$

$$\leq\frac{m^pb^p-a^p}{2(\alpha+1)}[|f'(a^p)|+m^p|f'(b^p)|](1-\frac{1}{2^\alpha})$$

Proof. Using Lemma 2.9 and m- convexity of $|f'|$, we have

$$|\frac{f(a^p)+f(m^pb^p)}{2}-\frac{p^\alpha.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^\alpha}.[{}^pI^\alpha_{mb-}f(a^p)+{}^pI^\alpha_{a+}f(m^pb^p)]|$$

$$\leq p(\frac{m^pb^p-a^p}{2})\int_0^1 t^{p-1}|[(1-t^p)^\alpha-t^{p\alpha}]|t^p.|f'(t^pa^p+m^p(1-t^p)b^p)|dt$$

$$\leq p(\frac{m^pb^p-a^p}{2})\int_0^1 t^{p-1}|[(1-t^p)^\alpha-t^{p\alpha}]|[t^p|f'(a^p)|+m^p(1-t^p)|f'(b^p)|]dt$$

$$\leq p(\frac{m^pb^p-a^p}{2})[\int_0^{1/p\sqrt{2}}t^{p-1}[(1-t^p)^\alpha-t^{p\alpha}][t^p|f'(a^p)|+m^p(1-t^p)|f'(b^p)|]dt$$

$$+\int_{1/p\sqrt{2}}^1 t^{p-1}[t^{p\alpha}-(1-t^p)^\alpha][t^p|f'(a^p)|+m^p(1-t^p)|f'(b^p)|]dt$$

$$=p(\frac{m^pb^p-a^p}{2})[\int_0^1 g(t)dt-2\int_0^{1/p\sqrt{2}}g(t)dt]$$

where $g(t)=t^{p-1}[t^{p\alpha}-(1-t^p)^\alpha][t^p|f'(a^p)|+m^p(1-t^p)|f'(b^p)|]$.

$$|\frac{f(a^p)+f(m^pb^p)}{2}-\frac{p^\alpha.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^\alpha}.[{}^pI^\alpha_{mb-}f(a^p)+{}^pI^\alpha_{a+}f(m^pb^p)]|$$

$$\leq p(\frac{m^pb^p-a^p}{2})[\frac{\alpha|f'(a^p)|}{p(\alpha+1)(\alpha+2)}-\frac{m^p\alpha|f'(b^p)|}{p(\alpha+1)(\alpha+2)}$$

$$-2[\frac{|f'(a^p)|}{2^{\alpha+2}p(\alpha+2)}+\frac{m^p|f'(b^p)|}{2^{\alpha+1}p(\alpha+1)}-\frac{m^p|f'(b^p)|}{p(\alpha+2)}$$

$$+\frac{|f'(a^p)|}{p}[\frac{1}{2^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{1}{2^{\alpha+2}(\alpha+1)}-\frac{1}{(\alpha+1)(\alpha+2)}]]]$$

$$\left|\frac{f(a^p)+f(m^pb^p)}{2}-\frac{p^\alpha.\Gamma(\alpha+1)}{2(m^pb^p-a^p)^\alpha}.[{}^pI^\alpha_{mb-}f(a^p)+{}^pI^\alpha_{a+}f(m^pb^p)]\right|$$
$$\leq \frac{m^pb^p-a^p}{2(\alpha+1)}[|f'(a^p)|+m^p|f'(b^p)|](1-\frac{1}{2^\alpha})$$

Corollary 2.11 Taking m=1 in the Theorem 2.10 we obtain Theorem 1.9 which is Theorem 2.5 of [4].

III. Some m-convex inequalities

Let $F(x)=f(x)+f(\frac{a}{m}+b-x)$. It is easy to show that if $f(x)$ is m-convex.Then function $F(x)$ is also m-convex.The function F has the following properties

1- $2f(\frac{a+mb}{2})=F(\frac{a+b}{2})$

2- $F(a)=F(b)=f(a)+f(b)$

3- $f(a)+mf(b)=\frac{F(a)+mF(b)}{2}$

Taking m=1 in 1,2 and 3, we have the result of [4] which was first discussed by M. Jeli and D.O.B Samet.

Theorem 3.1 Let f be an m-convex function on the interval $[a,mb]$ and $f\in L[a,mb]$. Then $F(x)$ is also integrable then the inequality hold

$$F(\frac{a+mb}{2})\leq \frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^\alpha}[{}^pI^\alpha_{a+/m}F(b)+{}^pI^\alpha_{b-}F(a/m)] \tag{14}$$
$$\leq \frac{F(a)+mF(b)}{2}$$

where $\alpha>0,\ p>0$.

Proof. As $f(x)$ is a m-convex function on $[a,mb]$, for $x,y\in[a,mb]$, $f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$. If $t=1/2$ then $f(\frac{x+my}{2})\leq \frac{f(x)+mf(y)}{2}$.

Let $x=ta+m(1-t)b,\ \ y=tb+\frac{(1-t)a}{m}$.

Then $2f(\frac{a+mb}{2})\leq f(ta+m(1-t)b)+f(tb+\frac{(1-t)a}{m})$.

Using the notation of $F(x)$

$$F(\frac{a+mb}{2})\leq F(tb+\frac{(1-t)a}{m}) \tag{15}$$

Multiplying both sides of (15) by

$$[\frac{(1-t)a}{m}+tb]^{p-1}\times[b^p-(tb+\frac{a}{m}(1-t))^p]^{\alpha-1} \tag{16}$$

and then integrating

$$\frac{(b^p-a^p/m^p)^\alpha}{p\alpha(b-a/m)}F(\frac{a+mb}{2})$$
$$\leq \int_0^1[\frac{(1-t)a}{m}+tb]^{p-1}\times[b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha-1}$$
$$\times[F(tb+\frac{(1-t)a}{m})dt]$$

$$=\int_{\frac{a}{m}}^b(u)^{p-1}\times(b^p-u^p)^{\alpha-1}\frac{F(u)}{b-\frac{a}{m}}du$$

$$=\frac{p^{\alpha-1}\Gamma(\alpha)}{b-\frac{a}{m}}({}^pI^\alpha_{a+/m}F(b))$$

$$F(\frac{a+mb}{2})\leq\frac{p^\alpha\Gamma(\alpha+1)}{(b^p-\frac{a^p}{m^p})^\alpha}({}^pI^\alpha_{a+/m}F(b)) \tag{17}$$

Similarly multiplying both sides of (15) by

$$[\frac{(1-t)a}{m}+tb]^{p-1}.[(\frac{(1-t)a}{m}+tb)^p-\frac{a^p}{m^p}]^{\alpha-1} \tag{18}$$

and integrating the result over $[0,1]$

$$F(\frac{a+mb}{2})\leq\frac{p^\alpha\Gamma(\alpha+1)}{(b^p-\frac{a^p}{m^p})^\alpha}({}^pI^\alpha_{b-}F(a/m)) \tag{19}$$

Now adding (17) and (19), we get

$$F(\frac{a+mb}{2})\leq\frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^\alpha}[{}^pI^\alpha_{a+/m}F(b)+{}^pI^\alpha_{b-}F(a/m)] \tag{20}$$

First inequality is established. Now we prove the second inequality. Since f is m-convex, then for $t\in[0,1]$

$$f(ta+m(1-t)b)+mf(tb+\frac{(1-t)a}{m})\leq f(a)+mf(b)$$

Using the notation of $F(x)$

$$F(tb+\frac{(1-t)a}{m})\leq\frac{F(a)+mF(b)}{2} \tag{21}$$

Multiplying both the sides of (21) with factor (16)

$$[\frac{(1-t)a}{m}+tb]^{p-1}\times[b^p-(tb+\frac{a(1-t)}{m})^p]^{\alpha-1}$$

and integrating it with respect of t over $[0,1]$, we get

$$\int_0^1[\frac{(1-t)a}{m}+tb]^{p-1}\times[b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha-1}F(tb+\frac{(1-t)a}{m})dt$$
$$\leq\frac{F(a)+mF(b)}{2}\int_0^1[\frac{(1-t)a}{m}+tb]^{p-1}\times[b^p-(tb+\frac{a(1-t)}{m})^p]^{\alpha-1}dt$$

which further gives

$$\frac{p^\alpha\Gamma(\alpha+1)}{(b^p-\frac{a^p}{m^p})^\alpha}({}^pI^\alpha_{a+/m}F(b))\leq\frac{F(a)+mF(b)}{2} \tag{22}$$

Similarly multiplying both the sides of (21) with the factor (18) and then integrating , we get

$$\frac{p^\alpha\Gamma(\alpha+1)}{(b^p-\frac{a^p}{m^p})^\alpha}[{}^pI^\alpha_{b-}F(\frac{a}{m})]\leq\frac{F(a)+mF(b)}{2} \tag{23}$$

Adding (22) and (23), we obtain

$$\frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^\alpha}[{}^pI^\alpha_{a+/m}F(b)+{}^pI^\alpha_{b-}F(a/m)]\leq\frac{F(a)+mF(b)}{2} \tag{24}$$

From (20) and (24)

$$F(\frac{a+mb}{2}) \leq \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^{\alpha}}[{}^pI^{\alpha}_{a+/m}F(b)+{}^pI^{\alpha}_{b-}F(a/m)] \leq \frac{F(a)+mF(b)}{2}$$

This completes the proof.

Corollary 3.2 Taking $m=1$ in Theorem 3.1, the inequality 14 become the result of [4]

$$F(\frac{a+b}{2}) \leq \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p-a^p)^{\alpha}}[{}^pI^{\alpha}_{a+}F(b)+{}^pI^{\alpha}_{b-}F(a)] \leq \frac{F(a)+F(b)}{2})$$

Remark 3.3 Theorem 3.1 is a generalization of Hermite-Hadamard inequality. If we take $m=1$ and $p \dagger 1$ in inequality of the Theorem 3.1.

Noticing that

$$\lim_{p\to 1} {}^pI^{\alpha}_{a+}F(b) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}F(t)dt = J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)$$

$$\lim_{p\to 1} {}^pI^{\alpha}_{b-}F(a) = \frac{1}{\Gamma(\alpha)}\int_a^b (t-a)^{\alpha-1}F(t)dt = J^{\alpha}_{b-}f(a)+J^{\alpha}_{a+}f(b)$$

Riemann-Liouville form of Hermite -Hadamard inequality [3] is obtained.

Corollary 3.4 Let f is a m-convex function on the interval $[a,mb]$ and $f \in L[a,mb]$ Then F(x) is also integrable and the following inequalities hold

$$F(\frac{a+bm}{2}) \leq \frac{\Gamma(\alpha+1)}{2(\ln(mb/a))^{\alpha}}[H^{\alpha}_{a+/m}F(b)+H^{\alpha}_{b-}F(a/m)] \leq \frac{F(a)+mF(b)}{2}$$

Proof. Using the inequality (14)

$$F(\frac{a+mb}{2}) \leq \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^{\alpha}}[{}^pI^{\alpha}_{a+/m}F(b)+{}^pI^{\alpha}_{b-}F(a/m)] \leq \frac{F(a)+mF(b)}{2}$$

Letting $p \to 0^+$ in equation (14) and noticing

$$\lim_{p\to 0^+} {}^pI^{\alpha}_{a+/m}F(b) = H^{\alpha}_{a+/m}F(b)$$

$$\lim_{p\to 0^+} {}^pI^{\alpha}_{b-}F(a/m) = H^{\alpha}_{b-}F(a/m)$$

So

$$F(\frac{a+bm}{2}) \leq \frac{\Gamma(\alpha+1)}{2(\ln(mb/a))^{\alpha}}[H^{\alpha}_{a+/m}F(b)+H^{\alpha}_{b-}F(a/m)] \leq \frac{F(a)+mF(b)}{2}$$

Remark 3.5 Taking $m=1$, we obtain the Hermite-Hadamard inequality for Hadamard fractional integral [4].

$$F(\frac{a+b}{2}) \leq \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^{\alpha}}[H^{\alpha}_{a+}F(b)+H^{\alpha}_{b-}F(a)] \leq \frac{F(a)+F(b)}{2}$$

Lemma 3.6 Let $f:[a,mb]\to R$ be a differentiable function on the interval (a,mb) with $a<mb$. If $f' \in L[a,mb]$ and $F' \in L[a,mb]$, then following equality holds

$$\frac{F(a/m)+F(b)}{2} - \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^{\alpha}}[{}^pI^{\alpha}_{a+/m}F(b)+{}^pI^{\alpha}_{b-}F(a/m)]$$

$$= \frac{(b-\frac{a}{m})}{2(b^p-\frac{a^p}{m^p})^{\alpha}}\int_0^1 \kappa(t)F'(tb+\frac{(1-t)a}{m})dt \quad (25)$$

where $\kappa(t) = [(tb+\frac{(1-t)a}{m})^p - \frac{a^p}{m^p}]^{\alpha} - [b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha}$

Proof. Take R.H.S

$$= \frac{(b-\frac{a}{m})}{2(b^p-\frac{a^p}{m^p})^{\alpha}}\int_0^1 \kappa(t)F'(tb+\frac{(1-t)a}{m})dt$$

Note that

$$I = \int_0^1 \kappa(t)F'(tb+\frac{(1-t)a}{m})dt$$

$$I = \int_0^1 [(tb+\frac{(1-t)a}{m})^p-\frac{a^p}{m^p}]^{\alpha}[F'(tb+\frac{(1-t)a}{m})]dt$$

$$-\int_0^1 [b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha}\ [F'(tb+\frac{(1-t)a}{m})]dt$$

therefore $I = I_1 + I_2$ (26)

where $I_1 = \int_0^1 [(tb+\frac{(1-t)a}{m})^p-\frac{a^p}{m^p}]^{\alpha}\ [F'(tb+\frac{(1-t)a}{m})]dt$

Let $u = tb+\frac{(1-t)a}{m}$. Then

$$I_1 = \frac{1}{(b-\frac{a}{m})}\int_{a/m}^b [u^p-\frac{a^p}{m^p}]^{\alpha}dF(u)$$

Integrating by parts, we obtain

$$I_1 = \frac{(b^p-\frac{a^p}{m^p})^{\alpha}}{(b-\frac{a}{m})}F(b) - \frac{p^{\alpha}\Gamma(\alpha+1)}{(b-\frac{a}{m})}.{}^pI^{\alpha}_{b-}F(a/m) \quad (27)$$

and $I_2 = -\int_0^1 [b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha}[F'(tb+\frac{(1-t)a}{m})]dt$

$$I_2 = -\frac{1}{(b-\frac{a}{m})}\int_0^1 [b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha}[F'(tb+\frac{(1-t)a}{m})][b-\frac{a}{m}]dt$$

Letting $u = tb+\frac{(1-t)a}{m}$, we have

$$I_2 = -\frac{1}{(b-\frac{a}{m})}\int_{a/m}^b [b^p-u^p]^{\alpha}dF(u)$$

Integrating by parts, we obtain

$$I_2 = \frac{(b^p-\frac{a^p}{m^p})^{\alpha}}{(b-\frac{a}{m})}F(a/m) - \frac{p^{\alpha}\Gamma(\alpha+1)}{(b-\frac{a}{m})}.{}^pI^{\alpha}_{a+/m}F(b) \quad (28)$$

From equation (27) and (28) put values of I_1 and I_2 in (26)

$$I = \frac{(b^p-\frac{a^p}{m^p})^{\alpha}}{(b-\frac{a}{m})}[F(a/m)+F(b)] - \frac{p^{\alpha}\Gamma(\alpha+1)}{(b-\frac{a}{m})}[.{}^pI^{\alpha}_{b-}F(a/m)+.{}^pI^{\alpha}_{a+/m}F(b)]$$

After simplification, the final result is obtained

$$\frac{F(a/m)+F(b)}{2} - \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^{\alpha}}[{}^pI^{\alpha}_{a+/m}F(b)+{}^pI^{\alpha}_{b-}F(a/m)]$$

$$= \frac{(b-\frac{a}{m})}{2(b^p-\frac{a^p}{m^p})^\alpha}\int_0^1 \kappa(t)F'(tb+\frac{(1-t)a}{m})dt$$

Corollary 3.7 Taking $m=1$ in Lemma 3.6

$$\frac{F(a)+F(b)}{2}-\frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-a^p)^\alpha}[{}^pI^\alpha_{a+}F(b)+{}^pI^\alpha_{b-}F(a)]$$

$$=\frac{(b-a)}{2(b^p-a^p)}\int_0^1\kappa(t)\ F'(tb+(1-t)a)dt$$

which is Lemma 3.4 of [4].

Theorem 3.8 Let $f:[a,mb]\to R$ be a differentiable mapping on the interval (a,mb) with $a<mb$, $f'\in L[a,mb]$ and $F'\in L[a,mb]$. Then we have

$$\left|\frac{F(a/m)+F(b)}{2}-\frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^\alpha}[{}^pI^\alpha_{a+/m}F(b)+{}^pI^\alpha_{b-}F(a/m)]\right|$$

$$\le \frac{(b-\frac{a}{m})}{2(b^p-\frac{a^p}{m^p})^\alpha}[|f'(b)|+\frac{|f'(a)|}{m}]\int_0^1|\kappa(t)|\,dt \tag{29}$$

where $\kappa(t)=[(tb+\frac{(1-t)a}{m})^p-\frac{a^p}{m^p}]^\alpha-[b^p-(tb+\frac{(1-t)a}{m})^p]^\alpha$

Proof. Using the notation of

$$F'(x)=f'(x)+f'(\frac{a}{m}+b-x)$$

By the m-convexity of $|f'(x)|$

$$|F'(tb+\frac{(1-t)a}{m}|=|f'(tb+\frac{(1-t)a}{m})+f'(\frac{a}{m}+b-tb-\frac{(1-t)a}{m})|$$

and therefore

$$|F'(tb+\frac{(1-t)a}{m})|\le|f'(b)|+\frac{|f'(a)|}{m} \tag{30}$$

Using (25) and (30)

$$\left|\frac{F(a/m)+F(b)}{2}-\frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^\alpha}[{}^pI^\alpha_{a+/m}F(b)+{}^pI^\alpha_{b-}F(a/m)]\right|$$

$$\le\frac{(b-\frac{a}{m})}{2(b^p-\frac{a^p}{m^p})^\alpha}\int_0^1|\kappa(t)|\,dt[|f'(b)|+\frac{|f'(a)|}{m}]$$

Hence

$$\left|\frac{F(a/m)+F(b)}{2}-\frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-\frac{a^p}{m^p})^\alpha}[{}^pI^\alpha_{a+/m}F(b)+{}^pI^\alpha_{b-}F(a/m)]\right|$$

$$\le\frac{(b-\frac{a}{m})}{2(b^p-\frac{a^p}{m^p})^\alpha}\int_0^1|\kappa(t)|\,dt[|f'(b)|+\frac{|f'(a)|}{m}]$$

Where $\kappa(t)=[(tb+\frac{(1-t)a}{m})^p-\frac{a^p}{m^p}]^\alpha-[b^p-(tb+\frac{(1-t)a}{m})^p]^\alpha$

Remark 3.9 If we take m=1 in Theorem 3.8, we obtain the following of [4]

$$\left|\frac{F(a)+F(b)}{2}-\frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-a^p)^\alpha}[{}^pI^\alpha_{a+}F(b)+{}^pI^\alpha_{b-}F(a)]\right|$$

$$\le\frac{(b-\frac{a}{m})}{2(b^p-\frac{a^p}{m^p})}\int_0^1\kappa(t)(|f'(a)|+|f'(b)|)$$

Remark 3.10 In Theorem 3.8, by letting $p\to1$ along with $m=1$

$$\lim_{p\to1}\int_0^1\kappa(t)dt=(b-a)^\alpha\int_0^1[t^\alpha-(1-t)^\alpha]dt$$

$$=(b-a)^\alpha[\int_0^{1/2}((1-t)^\alpha-t^\alpha)dt+\int_{1/2}^1(t^\alpha-(1-t)^\alpha)dt]$$

$$=\frac{2(b-a)^\alpha}{\alpha+1}[1-\frac{1}{2^\alpha}]$$

which is remark 3.6 of [4]. Using the fact $F(a)=F(b)=f(a)+f(b)$ and

$$\lim_{p\to1}{}^pI^\alpha_{a+}F(b)=\frac{1}{\Gamma(\alpha)}\int_b^a(b-t)^{\alpha-1}F(t)dt=J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)$$

$$\lim_{p\to1}{}^pI^\alpha_{b-}F(a)=\frac{1}{\Gamma(\alpha)}\int_b^a(t-a)^{\alpha-1}F(t)dt=J^\alpha_{b-}f(a)+J^\alpha_{a+}f(b)$$

Theorem 3.8 becomes as follows

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]\right|$$

$$\le\frac{(b-a)}{2(\alpha+1)}(1-\frac{1}{2^\alpha})[|f'(a)|+|f'(b)|]$$

which is [Theorem 3, 23].

Theorem 3.11 Let $f:[a,mb]\to R$ be an m-convex function with a, mb, $f\in L[a,mb]$ and $F\in L[a,mb]$. Then $F(x)$ is also m-convex function. If $g:[a,mb]\to R$ is non negative and integrable, then the following inequality holds

$$F(\frac{a+bm}{2})[{}^pI^\alpha_{a+/m}g(b)+{}^pI^\alpha_{b-}g(a/m)]$$

$$\le{}^pI^\alpha_{a+/m}(gF)(b)+{}^pI^\alpha_{b-}(gF)(a/m) \tag{31}$$

$$\le\frac{F(a)+mF(b)}{2}[{}^pI^\alpha_{a+/m}g(b)+{}^pI^\alpha_{b-}g(a/m)]$$

Proof. Since f is m-convex function on $[a,mb]$ and for all $t\in[0,1]$

$$2f(\frac{a+mb}{2})\le f(ta+m(1-t)b)+f(tb+\frac{(1-t)a}{m})$$

Using the notation of F(x)

$$F(\frac{a+mb}{2})\le F(tb+\frac{(1-t)a}{m}) \tag{32}$$

Multiplying both sides of (32) by

$$[\frac{(1-t)a}{m}+tb]^{p-1}\times[b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha-1}g(tb+\frac{(1-t)a}{m}) \tag{33}$$

and integrating

$$\frac{p^{\alpha-1}\Gamma(\alpha)}{(b-\frac{a}{m})}F(\frac{a+mb}{2})^{p}I^{\alpha}_{a+/m}g(b) \leq \int_0^1 F(tb+\frac{(1-t)a}{m})[\frac{(1-t)a}{m}+tb]^{p-1}$$

$$\times[b^p-(tb+\frac{(1-t)a}{m})^p]^{\alpha-1}g(tb+\frac{(1-t)a}{m})$$

$$=\int_{a/m}^{b}(gF)(x)(b^p-a^p)^{\alpha-1}\frac{(.x)^{p-1}dx}{(b-\frac{a}{m})}$$

$$=\frac{p^{\alpha-1}\Gamma(\alpha)}{(b-\frac{a}{m})}({}^{p}I^{\alpha}_{a+/m}gF(b))$$

$$F(\frac{a+mb}{2})({}^{p}I^{\alpha}_{a+/m}g(b)) \leq ({}^{p}I^{\alpha}_{a+/m}(gF)(b)) \quad (34)$$

Similarly multiplying both sides of inequality (32) by

$$[\frac{(1-t)a}{m}+tb]^{p-1}.[(\frac{(1-t)a}{m}+tb)^p-\frac{a^p}{m^p}]^{\alpha-1}g(tb+\frac{(1-t)a}{m}) \quad (35)$$

and integrating similarly we get

$$F(\frac{a+mb}{2})({}^{p}I^{\alpha}_{b-}g(a/m)) \leq ({}^{p}I^{\alpha}_{b-}(gF)(a/m)) \quad (36)$$

Adding inequalities (34) and (36)

$$F(\frac{a+mb}{2})[{}^{p}I^{\alpha}_{a+/m}g(b)+{}^{p}I^{\alpha}_{b-}g(a/m)]$$

$$\leq [{}^{p}I^{\alpha}_{a+/m}(gF)(b)+{}^{p}I^{\alpha}_{b-}(gF)(a/m)] \quad (37)$$

The first inequality is proved. Now for the second inequality. As f is m-convex function .Then for all $t\in[0,1]$

$$f(ta+m(1-t)b)+mf(tb+\frac{(1-t)a}{m}) \leq f(a)+mf(b)$$

Using the notation of $F(x)$

$$F(tb+\frac{(1-t)a}{m}) \leq \frac{F(a)+mF(b)}{2} \quad (38)$$

Multiplying both sides of (38) by factor (33) and integrating it, we get

$$\frac{p^{\alpha-1}\Gamma(\alpha)}{(b-\frac{a}{m})}[{}^{p}I^{\alpha}_{a+/m}(gF)(b)] \leq [\frac{F(a)+mF(b)}{2}]\frac{p^{\alpha-1}\Gamma(\alpha)}{(b-\frac{a}{m})}({}^{p}I^{\alpha}_{a+/m}g(b))$$

$$({}^{p}I^{\alpha}_{a+/m}(gF)(b)) \leq [\frac{F(a)+mF(b)}{2}]({}^{p}I^{\alpha}_{a+/m}g(b)) \quad (39)$$

Similarly multiplying both sides of (38) by the factor (35), and then integrating, we obtain

$$({}^{p}I^{\alpha}_{b-}(gF)(a/m)) \leq [\frac{F(a)+mF(b)}{2}]({}^{p}I^{\alpha}_{b-}g(a/m))$$

(40)

Adding (39) and (40)

$$[{}^{p}I^{\alpha}_{a+/m}(gF)(b)+{}^{p}I^{\alpha}_{b-}(gF)(a/m)]$$

$$\leq \frac{F(a)+mF(b)}{2}[{}^{p}I^{\alpha}_{b-}g(a/m)+({}^{p}I^{\alpha}_{a+/m}g(b)] \quad (41)$$

From (37) and (41), required result is obtained that is

$$F(\frac{a+mb}{2})[{}^{p}I^{\alpha}_{a+/m}g(b)+{}^{p}I^{\alpha}_{b-}g(a/m)]$$

$$\leq [{}^{p}I^{\alpha}_{a+/m}(gF)(b)+{}^{p}I^{\alpha}_{b-}(gF)(a/m)]$$

$$\leq \frac{F(a)+mF(b)}{2}[{}^{p}I^{\alpha}_{b-}g(a/m)+[{}^{p}I^{\alpha}_{a+/m}g(b)]$$

Remark 3.12 If $m=1$, Theorem 3.11 becomes the following result of [4]

$$F(\frac{a+b}{2})[{}^{p}I^{\alpha}_{a+}g(b)+{}^{p}I^{\alpha}_{b-}g(a)]$$

$$\leq [{}^{p}I^{\alpha}_{a+}(gF)(b)+{}^{p}I^{\alpha}_{b-}(gF)(a)]$$

$$\leq \frac{F(a)+F(b)}{2}[{}^{p}I^{\alpha}_{b-}g(a)+[{}^{p}I^{\alpha}_{a+}g(b)]$$

Remark 3.13 Theorem 3.11 is a generalization of Hermite-Hadamard-Fejer type inequalities of [13].

If we take $g(x)=1$ in Theorem 3.11, it becomes (14) of Theorem 3.1.

REFERENCES

[1] G. Abbas, G. Farid, Hadamard and Fejeˊr-Hadamard type inequalities for harmonically convex functions via generalized fractional integrals, J. Anal., 25(1), 2017, pp 107-119.

[2] L. Ali, T. Afzal,M. Aslam, Y.A. Khan, Hermite-Hadamard Inequalities for Riemann-Liouville Fractional Integrals for η-m-convex functions, Journal of Xi'an University of Architecture and Technology, 12(5), 2020, pp 2118-2131.

[3] R. Almeida, A.B. Malinowska, T.Odzijewicz,Fractional differential equations with dependence on Caputo- Katugampola derivative, Journal of Computional and non linear Dynamics (2016) DOI:10.115/1.4034432 at arXiv:1607.06913.

[4] H. Chen, U.N.Katugampola,Hermite-Hadamard and Hermite-Hadamard Fejer type inequalities for generalized fractional integrals, Math.Anal.Appl., 446, 2017, pp 1274-1291.

[5] Z. Dahmani, L. Tabharit, S. Taf, New generalizations of Gruss inequality using Riemann-Liouville fractional integrals, Bull. Math. Anal. Appl.2 (3), 2010, pp 93-99.

[6] S.S. Dragomir, C.E.M Pearce, Selcted Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs,Victoria University 2000.

[7] S.S. Dragomir and G.H. Toader, Some Inequalities for m-Convex Functions, Studia Univ. Babes-Bolyai, Math., 38(1), 1993,pp 21–28.

[8] G. Farid, AU Rehman, Bushra Tariq, A. Waheed, On Hadamard Type Inequalities For m-convex Functions VIA Fractional Integrals, Journal of Inequalities and Special Functions, 7, 2016, pp 150-155.

[9] G. Farid, A Treatment of the Hadamard inequality due to m-convexity via generalized fractional integral, J. Fract. Calc. Appl., 9(1), 2018, pp 8-14.

[10] G. Farid, Hadamard and Fejeˊr-Hadamard inequalities for generalized fractional integral involving special functions, Konuralp J. Math.,4(1) 2016, pp 108-113.

[11] M.E. Gordji, M.R Delavar and S.S Dragomir, Some inequalities related to η-convex function, Preprint Rgmia Res. Rep. Coll., 2015, pp 1–14.

[12] M. Iqbal, M.Iqbal Bhatti and Nazeer K,Generalization of inequalities analogous to Hermite-Hadamard inequality VIA fractional integrals Bull.Koren Math 52(3) , 2015, pp 707-716.

[13] I. Iscan, S. Wu ,Hermite Hadamard -Fejer type inequalities for convex functions VIA Fractional Integrals,Preprint arXiv (2014)

[14] M. Jleli, D. O'Regan, B. Samet,On Hermite Hadamard Type Inequalities VIA generalized Fractional Integrals,Turkish J. Math, 40(6), 2016, pp 1221-1230.

[15] M.S. Johar, A. Laptev, Convex function and their applications, M3R Project 2013

[16] S. M. Kang, G. Farid, W. Nazeer and S. Mehmood, (h, m)-convex functions and associated fractional Hadamard and Fejer-Hadamard inequalities via an extended generalized Mittag-Leffler function, J. Inequal. Appl.,78, 2019.

[17] U.N. Katugampola,A new approach to generalized fractional derivatives, Bulletin of Mathematical Analysis and Applications, 6(4), 1-15, 2014.

[18] U.N. Katugampola, New approach to a generalized fractional integrals Appl Math.Comput, 218(3), 2011, pp 860-865.
[19] H. Kavurmaci,M. Avci and M.E. Ozdemir,New inequalities of Hermite-Hadamard type for convex functions with applications,
[20] T. Lara, E. Rosales and J.L. Sanchez,New properties of m-convex functions, International Journal of Mathematical Analysis, 9(15), 2015, pp 735-742. (http://dx.doi.org/10.12988/ijma.2015.412389)
[21] M.A Noor, K.I Noor and M.U. Awan Convexity and Integral Inequalities,Appl,Math,Inf.Sci., 9(1), 2015, pp 233-243.
[22] A.W. Roberts, D.E.Varberg, Convex Functions Academic Press, New York and London 1973. Journal of Inequalities and Applications 2011.
[23] M.Z. Sarikaya ,E. Set, H. Yaldiz, N. Bashak, Hermite-Hadamard's inequalities for fractional integrals related fractional inequalities, Mathematical and Computer Modeling, 57, 2013, pp 2402-2407.
[24] G. H. Toader, Some generalizations of the Convexity, Proc. Colloq. Approx. Optim. Cluj-Naploca (Romania), 1984, pp 329–338.
[25] G.H. Toader, Hierarchies of convexity of functions, 2015
[26] S. Ullah, G. Farid , K. A. Khan, A. Waheed and S. Mehmood, Generalized fractional inequalities for quasi-convex functions, Adv. Difference Equ., 2019, 2019.
[27] N. Wheeler, Construction and physical application of the fractional calculus, 1997.

AUTHORS

First Author – Tehmina Afzal, MPhil, Department of Mathematics, Govt. College University Lahore, Pakistan
afzal.tehmina37@gmail.com
Second Author – Liaqat Ali, MPhil, Assistant Professor of Mathematics, Govt. MAO College Lahore, Pakistan
rehmani.pk786@gmail.com, rehmani_lhr@yahoo.com
Third Author – Muhammad Aslam, PhD, Professor of Mathematics, Govt. College University Lahore, Pakistan
Aslam298@gcu.edu.pk
Fourth Author – Yaqoub Ahmed Khan, Assistant Professor of Mathematics, GIC Lahore, Pakistan
yaqoubahmedkhan@gmail.com
Correspondence Author – Liaqat Ali
rehmani.pk786@gmail.com, rehmani_lhr@yahoo.com
+923004557189