Lie Symmetries and Classifications of (2 + 1)-dimensional Generalized Gardner Equation with Damping Term

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Abstract

In this paper, We are consider a (2+1)-dimensional generalized Gardner equation with damping term is subjected to the Lie symmetry group method. Classification of its symmetry algebra into one- and two-dimensional subalgebras is carried out in order to facilitate its reduction systematically to (1+1)-dimensional partial differential equations and then to first order ordinary differential equations.
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Introduction

The study of non-linear evolution equations, which describes a large variety of physical phenomena is an important area of research that has gained a lot of attention in the past few decades [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]. There are different situations in which it is necessary to consider dual non-linear terms. For example, in a density stratified ocean, where internal gravity waves are observed, the single non-linear term does not correctly model the shallow water waves.

Basically, the KdV equation with dual-power law with non-linearity is the Gardner equation.

$$u_t + pu_x + quu_x + ru^2u_x + su_{xxx} = 0, \tag{1}$$

where p, q, r and s are arbitrary non-zero constants. Krishnan et al. [?, ?] have studied the dynamics of solitary waves governed by this equation. They used the mapping method to carry out the integration of the equation and studied the perturbed Gardner equation, from which the fixed point of the soliton width was obtained and also carried out the integration of the perturbed Gardner equation with the aid of He's [?] semi-inverse variational principle. Johnpillai et al. [?] have shown that the Benjamin-Bona-Mahoney (BBM) equation with power law non-linearity can be transformed by a point transformation to the combined KdV-mKdV equation, which is also known as the Gardner equation,

$$u_t + auu_x + bu^2u_x + u_{xxx} = 0, \tag{2}$$

where a and b are arbitrary non-zero constants. After that, they have studied the combined KdV-mKdV equation from the Lie group-theory. The generators of the combined KdV-mKdV equation were derived the symmetry reductions by Lie point symmetry. Also, by the Lie point symmetries of the equations were obtained from a number of exact group-invariant solutions for that equation. Some of the results are especially interested in physical and mathematical sciences.

In this paper, we consider the (2 + 1)- dimensional generalized Gardner equation with damping term,

$$u_t + (2\delta u - 3\sigma u^2)u_x + u_{xxx} + u_{yy} + u = 0. \quad (3)$$

Thus, we investigate the symmetries and reductions of (??).

We showed that the (??), admits a symmetry group and determine the corresponding Lie algebras, then classifies the one- and two- dimensional subalgebras of the symmetry algebras of (??), in order to reduce (1+1)-dimensional partial differential equations and then to ordinary differential equations. We establish that the symmetry generators form a closed Lie algebra and this allowed us to successively reduce (??) to (1+1)-dimensional partial differential equation and ordinary differential equation with the help of two dimensional abelian and solvable non-abelian subalgebras.

Lie Symmetry Group and Lie Algebra of (2+1)-dim Generalized Gardner Equation with Damping Term

If (??) is invariant under a one parameter Lie group of point transformations,

$$\begin{aligned} x^* &= x + \epsilon\xi(x,y,t;u) + O(\epsilon^2)\ , && (4)\\ y^* &= y + \epsilon\eta(x,y,t;u) + O(\epsilon^2)\ , && (5)\\ t^* &= t + \epsilon\tau(x,y,t;u) + O(\epsilon^2)\ , && (6)\\ u^* &= u + \epsilon\phi(x,y,t;u) + O(\epsilon^2), (7) \end{aligned}$$

with the infinitesimal generator $\xi,\ \eta,\ \tau\ and\ \phi$. Then, the third prolongation $pr^{(3)}V$ of the corresponding vector field,

$$V = \xi(x,y,t;u)\partial_x + \eta(x,y,t;u)\partial_y + \tau(x,y,t;u)\partial_t + \phi(x,y,t;u)\partial_u, \quad (8)$$

satisfies

$$pr^{(3)}V\Omega(x,y,t;u)|_\Omega(x,y,t;u) = 0. \quad (9)$$

In order, to determine the four infinitesimal ξ, η, τ and ϕ. Then, the third order prolongation V is given by the formula,

$$\begin{aligned} V^{(3)} &= v + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{xt}\frac{\partial}{\partial u_{xt}}\\ &\quad + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxy}\frac{\partial}{\partial u_{xxy}}\\ &\quad + \phi^{xxt}\frac{\partial}{\partial u_{xxt}}. \end{aligned} \quad (10)$$

In the above expression, every co-efficient of the prolonged generator is a function of x, y, t and u can be determined by the formulae,

$$\begin{aligned} \phi^i &= D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i}, && (11)\\ \phi^{ij} &= D_iD_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij}, && (12)\\ \phi^{ijk} &= D_iD_jD_k(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijk} + \eta u_{y,ijk} + \tau u_{t,ijk}, && (13) \end{aligned}$$

where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates. To proceed with reductions of (??), we now use symmetry criterion for the partial differential equation. For the given equation, this criterion is expressed by the formula,

$$V^{(3)}\left(u_t+u_x+u_y+uu_x+uu_y-u_{xxt}-u_{yyt}\right)=0,$$

whenever,

$$\phi^t+\phi^x+\phi^y+\phi\phi^x+\phi\phi^y-\phi^{xxt}-\phi^{yyt}=0. \tag{14}$$

Using (?? - ??), we introduced the following quantities,

$$\begin{aligned}
\phi^x &= D_x\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)+\xi u_{xx}+\eta u_{yx}+\tau u_{tx}\\
&= \phi_x+\left(\phi_u-\xi_x\right)u_x-\eta_x u_y-\tau_x u_t-\xi_u u_x^2-\eta_u u_x u_y-\tau_u u_x u_t,\\
\phi^{xx} &= D_x D_x\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)+\xi u_{xxx}+\eta u_{yxx}+\tau u_{txx}\\
&= \phi_x x+\left(2\phi_{xu}-\xi_{xx}\right)u_x-\eta_{xx}u_y-\tau_{xx}u_t+\left(\phi_u-2\xi_x\right)u_{xx}-2\eta_x u_{xy}\\
&\quad -2\tau_x u_{xt}+\left(\phi_{uu}-2\xi_{ux}\right)u_x^2-2\eta_{ux}u_x u_y-2\tau_{xu}u_x u_t-\xi_{uu}u_x^3-3\xi_u u_x u_{xx}\\
&\quad -\eta_{uu}u_x^2 u_y-\tau_{uu}u_x^2 u_t-2\eta_u u_x u_{xy}-\eta_u u_{xx}u_y-\tau_u u_{xx}ut-2\tau_u u_x u_x t,\\
\phi^{yy} &= D_y D_y\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)+\xi u_{xyy}+\eta u_{yyy}+\tau u_{tyy}\\
&= \phi_{yy}-\xi_{yy}u_x+\left(2\phi_{yu}-\eta_{yy}\right)u_y-\tau_{yy}u_t-2\xi_y u_{xy}+\left(\phi_u-2\eta_y\right)u_{yy}\\
&\quad -2\xi_{yu}u_x u_y-2\tau_{yu}u_y u_t+\left(\phi_{uu}-2\eta_{yu}\right)u_y^2-2\tau_y u_{yt}-2\xi_u u_y u_{xy}-3\eta_u u_y u_{yy}\\
&\quad -\xi_{uu}u_y^2 u_x-\xi_u u_{yy}u_x-\eta_{uu}u_y^3-2\tau_u u_y u_{yt}-\tau_u u_{yy}u_t-\tau_u u u_y^2 u_t,\\
\phi^t &= D_t\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)+\xi u_{x,t}+\eta u_{y,t}+\tau u_{t,t}\\
&= \phi_u+u_t\left(\phi_u-\tau_t\right)-\tau_u u_t^2-\xi_u u_t u_x-\xi_t u_x-\eta_u u_t u_y-\eta_u u_y.
\end{aligned}$$

Substituting all the above formulae in (??) and then compare the co-efficients of various monomials in derivatives of "u". This yields the determining equation.
They are as follows:

$$\begin{aligned}
\left(\xi_1\right)_u &= 0,\\
\left(\xi_2\right)_u &= 0,\\
\left(\xi_3\right)_u &= 0,\\
\left(\phi_1\right)_{u,u} &= 0,\\
\left(\xi_1\right)_y &= 0,\\
\left(\xi_3\right)_y &= 0,\\
\left(\xi_2\right)_x &= 0,\\
\left(\xi_3\right)_x &= 0,\\
\phi_1+2u\phi_1-\left(\xi_1\right)_t+2\left(\xi_1\right)_x+2u\left(\xi_1\right)_x+2u^2\left(\xi_1\right)_x &\\
-\left(\xi_1\right)_{x,x,x}+3\left(\phi_1\right)_{x,x,u} &= 0,
\end{aligned}$$

$$\begin{aligned}\phi_1 + 3u\left(\xi_1\right)_x + \left(\phi_1\right)_t - u\left(\phi_1\right)_u + \left(\phi_1\right)_x + u\left(\phi_1\right)_x x & \\ +u^2\left(\phi_1\right)_x + \left(\phi_1\right)_{y,y} + \left(\phi_1\right)_{x,x,x} &= 0, \\ -\left(\xi_1\right)_{x,x} + \left(\phi_1\right)_{x,u} &= 0, \\ 3\left(\xi_1\right)_x - 2\left(\xi_2\right)_y &= 0, \\ 3\left(\xi_1\right)_x - \left(\xi_3\right)_t &= 0, \\ -\left(\xi_2\right)_t - \left(\xi_2\right)_{y,y} + 2\left(\phi_1\right)_{y,u} &= 0. \end{aligned} \tag{15}$$

After solving the above determining equations, we obtained the following infinitesimals.
Infinitesimals:

$$\xi = k_3, \tag{16}$$
$$\eta = k_2, \tag{17}$$
$$\tau = k_1, \tag{18}$$
$$\phi = 0. \tag{19}$$

Now, we construct the symmetry generators corresponding to each of the constants involved. Totally there are four generators given by

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t}, \\ V_2 &= \frac{\partial}{\partial y}, \\ V_3 &= \frac{\partial}{\partial x}. \end{aligned} \tag{20}$$

Thus, we can construct a one-parameter group of invariants $g_i(\epsilon)$, generated by the vector V_i, where $i = 1, 2$ *and* 3 are given as follows,

$$\begin{aligned} g_1(\epsilon) : (x, y, t; u) &\rightarrow (x, y, t + \epsilon, u), \\ g_2(\epsilon) : (x, y, t; u) &\rightarrow (x, y + \epsilon, t, u), \\ g_3(\epsilon) : (x, y, t; u) &\rightarrow (x + \epsilon, y, t, u). \end{aligned}$$

Hence, g_1 is the time translations. Also, g_2 and g_3 are the space-invariant of the equation. Next, we construct the commutator table with the symmetric generators. It is shown below:

Commutator Table

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	0
V_3	0	0	0

In the above table are shown the commutator relations of the Lie algebra L, determined by the vectors V_1, V_2 *and* V_3. For the commutator table of three-dimensional Lie Algebra V_i is a $(3 \otimes 3)$ table whose $(i, j)^{th}$ entry expresses the Lie Bracket $[V_i, V_j]$ given by the above Lie Algebra L. Therefore, the commutator table is skew-symmetric and the diagonal elements all vanish. Also, the related structure constants can be calculated from the above table are as follows and the Lie algebra L is solvable. Thus, the radical of G is $R = < V_1, V_2, V_3 >$.

Symmetry Reductions of (2+1)-dim Generalized Gardner Equation by one-dimensional Subalgebras

In this section, we derive the reductions of (??) to a partial differential equation with two independent variables and ordinary differential equations. These are three one-dimensional Lie subalgebras are given below:

$$L_{s,1} = \{V_1\}\ ,\ L_{s,2} = \{V_2\}\ and\ L_{s,3} = \{V_3\}$$

and corresponding to each one-dimensional subalgebras we may reduce (??) to a partial differential equation with two independent variables.

Case 1:
The Subalgebra $L_{s,1} = V_1 = \partial_t$.
The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}\ .$$

Thus, we integrate the characteristic equation to get their corresponding similarity variables,

$$x = a; y = b; u = w(a, b). \tag{21}$$

Using these similarity variables in (??) can be transformed in the form

$$(2\delta w - 3\sigma w^2)w_a + w_{aaa} + w_{bb} + w = 0. \tag{22}$$

Case 2:
The Subalgebra $L_{s,2} = V_2 = \partial_y$.
The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}\ .$$

Thus, we integrate the above characteristic equation to find the similarity variables, which shown below:

$$x = a; t = b; u = w(a, b). \tag{23}$$

Using these similarity variables in (??) can be transformed in the form

$$(2\delta w - 3\sigma w^2)w_a + w_{aaa} + w_b + w = 0. \tag{24}$$

Case 3:
The Subalgebra $L_{s,3} = V_3 = \partial_x$.
The characteristic equation associated with V_3 is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}\ .$$

Hence, we can integrate the characteristic equation to obtain the similarity variables,

$$y = a; t = b; u = w(a, b). \tag{25}$$

Using these similarity variables in (??) can be transformed in the form

$$w_{aa} + w_b + w = 0. \tag{26}$$

Table 1

$S.No.$	$Vector Field$	$Similarity\ Variables$	$Similarity\ Reductions$
1	$V_1 = \partial_t$	$x = a; y = b; u = w(a, b)$	$(2\delta w - 3\sigma w^2)w_a + w_{aaa} + w_{bb} + w = 0$
2	$V_2 = t\partial_y$	$x = a; t = b; u = w(a, b)$	$(2\delta w - 3\sigma w^2)w_a + w_{aaa} + w_b + w = 0$
3	$V_3 = -\partial_x$	$y = a; t = b; u = w(a, b)$	$w_{aa} + w_b + w = 0$

Symmetry Reductions of (2+1)-dim Gardner Equation by two-dimensional Abelian Subalgebras

Further reductions to ordinary differential equations are associated with two-dimensional subalgebras. It is evident from the commutator table that there are three two-dimensional solvable abelian subalgebras namely,

$$L_{A,1} = \{V_1, V_2\}, L_{A,2} = \{V_1, V_3\}\ , and\ L_{A,3} = \{V_2, V_3\}.$$

There is no Non-abelian subalgebra in the commutator table.

Case : 1

The Subalgebra $L_{A,1} = \{V_1, V_2\}$.

In this case, we find that the given generators commute $[V_1, V_2] = 0$. Thus, either of V_1 or V_2 can be used to start the reduction procedure. For our purpose, we begin the reduction with V_1. In this case, (**??**) is reduced to the partial differential equation (**??**).
Now, we express V_1 in terms of the similarity variables defined in (**??**) as,

$$V_2^* = \partial_b. \tag{27}$$

The associated characteristic equation is

$$\frac{da}{0} = \frac{db}{1} = \frac{dw}{0}.$$

Integrating the above equation as before yields to the new variables $a = \zeta$, $F(\zeta) = w$. As a result (**??**) reduces to,

$$F_{\zeta\zeta\zeta} + (2\delta F - 3\sigma F^2)F_\zeta = 0. \tag{28}$$

Case : 2

The Subalgebra $L_{A,2} = \{V_1, V_3\}$.

In this case, we find that the given generators commute $[V_1, V_3] = 0$. Thus, either of V_1 or V_3 can be used to start the reduction procedure. For our convenience, we begin the reduction with V_1. In this case, (**??**) is reduced to the partial differential equation (**??**).
Now, we express V_1 in terms of the similarity variables defined in (**??**) as

$$V_3^* = \partial_a. \tag{29}$$

The associated characteristic equation is,

$$\frac{da}{1} = \frac{db}{0} = \frac{dw}{0}.$$

Integrating the above equation as before yields to the new variables $b = \zeta$, $F(\zeta) = w$. As a result (**??**) reduces to

$$F + F_{\zeta\zeta} = 0. \tag{30}$$

Case : 3
The Subalgebra $L_{A,3} = \{V_2, V_3\}$.

In this case, we find that the given generators commute $[V_2, V_3] = 0$. Thus, either of V_2 or V_3 can be used to start the reduction procedure. For our purpose, we begin the reduction with V_2. In this case, (**??**) is reduced to the partial differential equation (**??**).
Now, we express V_3 in terms of the similarity variables defined in (**??**) as

$$V_3^* = \partial_a. \tag{31}$$

The associated characteristic equation is

$$\frac{da}{1} = \frac{db}{0} = \frac{dw}{0}.$$

Integrating the above equation as before yields to the new variables $\zeta = b$, $F(\zeta) = w$. As a result (**??**) reduces to,

$$F + F_{\zeta} = 0. \tag{32}$$

Table 2

$S.No.$	$LieBracket$	$Similarity\ Variables$	$Similarity\ Reductions$
1	$[V_1, V_2] = 0$	$a = \zeta; w = F(\zeta)$	$F_{\zeta\zeta\zeta} + (2\delta F - 3\sigma F^2)F_\zeta = 0$
2	$[V_1, V_3] = 0$	$b = \zeta; w = F(\zeta)$	$F_{\zeta\zeta} + F = 0$
3	$[V_2, V_3] = 0$	$b = \zeta; w = F(\zeta)$	$F + F_\zeta = 0$

Conclusion

We determined (2+1)-dimensional Generalized Gardner equation with damping term is subjected to Lie Classical Method. The Gardner equation (**??**) admits a three-dimensional symmetry group. We established that the symmetry generators form a closed Lie algebra and the classifications of symmetry algebras of (**??**) into one- and two-dimensional abelian subalgebras are carried out. Finally, from the Table 1 and 2, we obtained, the classification of systematic reductions to (1+1)-dimensional partial differential equation and then to first or second order ordinary differential equations are performed through one-dimensional and two-dimensional solvable abelian subalgebras.

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Conflict of Interests:

The authors have declared that no Conflict of Interest exists.

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