Goodness of Fit Tests for Generalized Quasi Lindley Distribution

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Abstract- In this article, several goodness of fit tests for the generalized quasi Lindley distribution are suggested based on the commonly used simple random sampling (SRS) and ranked set sampling (RSS) methods. These tests include the Kolmogorov-Smirnov test, Anderson-Darling test, Cramer-von Mises test, and Zhang test. The power of the tests and the critical values are obtained based on SRS and RSS schemas for various alternatives and different sample sizes. A comparison study is performed to study the effectiveness goodness of fit tests based on RSS relative to its counterparts in SRS based on the same number of measured units. An application of real data set of 72 guinea pigs infected with virulent tubercle bacilli is given for illustration. The results indicate that the RSS tests performs better than the SRS counterparts.

***Keywords*:** Goodness of fit tests, Maximum likelihood estimation, Power of test, Ranked set sampling, Significance level.

1 Introduction

Ranked Set Sampling (RSS) method was introduced by McIntyre (1952) as an alternative method for collecting data to Simple Random Sampling (SRS). It was proposed to improving precision in estimation of a population mean. The RSS strategy can be described as follow:

1. Draw a SRS of size h^2 from the target population. Randomly sperate them it into h sets each of size h. h is called the set size.
2. Rank the units within each set of size h in ascending order.
3. Chose the i^{th} ranked units from the i^{th} set, $i = 1, 2, ..., h$ for actual quantification.
4. Repeat the steps (1)–(3), r times (cycles) if needed to get a RSS of size $M = rm$.

The SRS estimator of the population mean from a sample of size h is given by

$$\bar{X}_{SRS} = \frac{1}{h}\sum_{i=1}^{h} X_{ih} \text{ ,with variance } \operatorname{Var}\left(\bar{X}_{SRS}\right) = \frac{\sigma^2}{h}.$$

The estimator of the population for a RSS of size h is given by

$$\bar{X}_{RSS} = \frac{1}{h}\sum_{i=1}^{h} X_{i[i])} \text{ with variance}$$
$$\text{Var}\left(\bar{X}_{RSS}\right) = \frac{1}{h^2}\sum_{i=1}^{h}\text{Var}\left(X_{i[i]}\right) = \frac{\sigma^2}{h} - \frac{1}{h^2}\sum_{i=1}^{h}\left(\mu_{[i]} - \mu\right)^2$$

If $X \sim f(x)$, mean μ and variance σ^2. Let X_{i1},X_{i2},...,X_{ih} $(i = 1, 2, ..., h)$ be a random sample with probability density function $f(x)$. The quantified units are denoted by $\{X_{[i]j}, i = 1, \ldots, h; j = 1, 2, \ldots, r\}$, where $X_{[i]j}$ is the i^{th} largest unit in a set of size h in the j^{th} cycle. Let $X_{[i]} \sim F_{[i]}(x)$. Then

$$\begin{aligned} F_{[i]}(x) &= \frac{1}{B(i, h-i+1)}\int_0^{F(x)} u^{i-1}(1-u)^{j-i}du \\ &= B_{[i]}\left[F(x)\right], -\infty < x < \infty, \end{aligned}$$

where $B_{[i]} \sim B(i, h-i+1)$, $B(\alpha, \beta) = \dfrac{\Gamma(\alpha).\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, $\Gamma(\alpha) = (\alpha-1)!$, $\mu_{[i]} = \int\limits_{-\infty}^{\infty} x f_{[i]}(x)dx$, and $\sigma^2_{[i]} = \int_{-\infty}^{\infty}\left(x - \mu_{[i]}\right)^2 f_{[i]}(x)dx$.

Various modifications of the RSS are suggested in the literature and it is considered in wide applications in engineering, medicine, environmental studies, agriculture, etc. A new RSS design is known as double RSS is suggested by Al-Saleh and Al-Kadiri (2000), multistage RSS by Al-Saleh and Al-Omari (2002), median RSS by Muttlak (1997), neoteric RSS by Zamanzade and Al-Omari (2016), extreme RSS by Samawi et al. (1996), L RSS by Al-Nasser (2007).

For more about RSS, you can see Al-Omari and Bouza (2014), Al-Hadhrami and Al-Omari (2014), Haq et al. (2015), Santiago et al (2016), Haq et al (2016). In the literature, some authors apply the goodness of fit based on entropy and empirical distribution function using RSS design such as Al-Omari and Haq (2012) for the inverse Gaussian distribution, Al-Omari and Zamanzade (2016) for Rayleigh distribution, Al-Omari and Haq (2016) tests for the inverse Gaussian and Laplace distributions using pair ranked set sampling, Al-Omari and Zamanzade (2017) for Laplace distribution. For more details on goodness of fit and entropy you can see Al-Omari (2014), Al-Omari (2015), Al-Omari (2016),Al-Omari and Zamanzade (2018),Ebrahimi et al (1994),Van Es (1992). in this work, we give some goodness of fit test using SRS and RSS design for Generalized Quasi Lindley distribution (GQLD) proposed by Benchiha and Al-Omari (2021).

The layout of this paper is as follow. The GQLD is presented with some of its properties as moments and reliability functions in Section 2. Different goodness of fit tests are discussed in Section 3 using SRS and RSS schemes. Section 4 concerns with a simulation study to compare the performances of the suggested tests based on SRS and RSS. Finally, our conclusion is presented in Section 5 with recommendations for future works.

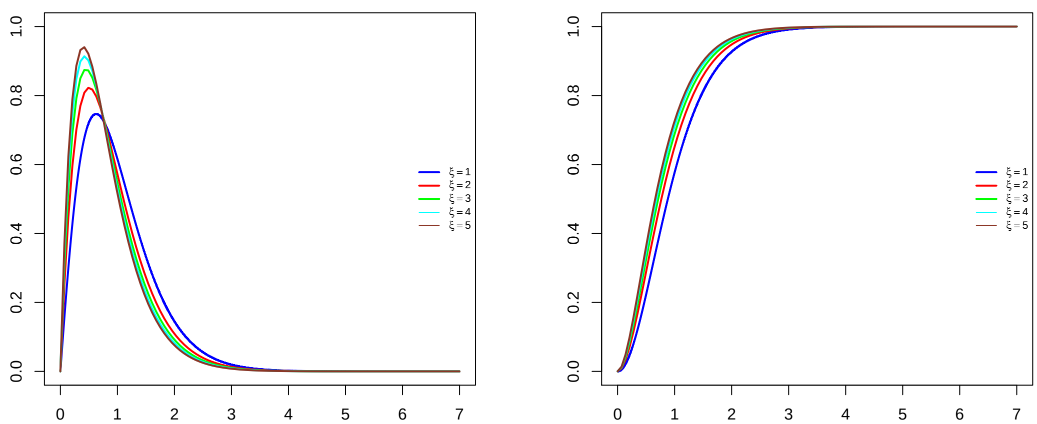
2 Generalized Quasi Lindley distribution

Generalized Quasi Lindley distribution (GQLD) is proposed by Benchiha and Al-Omari (2021) as a new lifetime distribution. The GQLD is a sum of two independent quasi Lindley distributed random variables. Let X follows a GQLD with parameters ψ and ξ, then the probability density function (pdf)of X is given by

$$g_{GQLD}(x;\psi,\xi) = \frac{\xi^2\left(\frac{\psi^2 x^3}{6} + \xi\psi x^2 + \xi^2 x\right)\mathrm{e}^{-\psi x}}{(\xi+1)^2};\ x \geq 0,\ \xi > -1,\ \psi > 0, \tag{1}$$

and its cumulative distribution function is defined as:

$$G_{GQLD}(x;\psi,\xi\ ) = 1 - \frac{\left(\psi^3 x^3 + 3\left(2\xi+1\right)\psi^2 x^2 + 6\left(\xi+1\right)^2\left(\psi x+1\right)\right)\mathrm{e}^{-\psi x}}{6\left(\xi+1\right)^2}. \tag{2}$$



(a) (b)

Figure 1: The pdf and cdf of the GQLD with Different values of ξ when $\psi = 3$.

It is clear that the distribution is asymmetric to the right. The first two moments of X are:

$$E(X) = \frac{6\left(\xi^2+1\right)+6\xi+6}{3\left(\xi+1\right)^2\psi}, \tag{3}$$

$$E(X^2) = \frac{6\left(\xi+1\right)^2 + 2\left(6\xi+5\right)+4}{\left(\xi+1\right)^2\psi^2}, \tag{4}$$

Therefore, the variance of the GQLD distribution is given by:

$$V(X) = E(X^2) - (E(X))^2 = \frac{2\left(\xi^2 + 4\xi + 2\right)}{\left(\xi + 1\right)^2 \psi^2}. \tag{5}$$

The corresponding reliability and hazard functions of the GQLD distribution are given, respectively by:

$$R_{GQLD}(x;\psi,\xi) = \frac{\left(\psi^3 x^3 + 3\left(2\xi + 1\right)\psi^2 x^2 + 6\left(\xi + 1\right)^2 \psi x + 6\left(\xi + 1\right)^2\right) \mathrm{e}^{-\psi x}}{6\left(\xi + 1\right)^2}; x > 0,\ \xi > -1,\ \psi > 0, \tag{6}$$

$$H_{GQLD}(x;\psi,\xi) = \frac{6\psi^2\left(\frac{\psi^2 x^3}{6} + \xi\psi x^2 + \xi^2 x\right)}{\psi^3 x^3 + 3\left(2\xi + 1\right)\psi^2 x^2 + 6\left(\xi + 1\right)^2 \psi x + 6\left(\xi + 1\right)^2}. \tag{7}$$

The reversed hazard rate and odds functions for the GQLD distribution, respectively, are defined as

$$RH_{GQLD}(x;\psi,\xi) = \frac{\psi^2 x\left(\psi^2 x^2 + 6\xi\psi x + 6\xi^2\right)}{6\left(\xi + 1\right)^2 \mathrm{e}^{\psi x} - \psi x\left(\psi x\left(\psi x + 6\xi + 3\right) + 6\left(\xi + 1\right)^2\right) - 6\left(\xi + 1\right)^2}. \tag{8}$$

and

$$O_{GQLD}(x;\psi,\xi) = \frac{6\left(\xi + 1\right)^2 \mathrm{e}^{\psi x}}{\psi^3 x^3 + 3\left(2\xi + 1\right)\psi^2 x^2 + 6\left(\xi + 1\right)^2 \psi x + 6\left(\xi + 1\right)^2} - 1. \tag{9}$$

3 Test statistics

In this section, we will discuss the suggested goodness of fit tests based on SRS and RSS methods.

3.1 Using SRS

Let $X_1, X_2, \ldots, X_m$ be a random sample from GQLD and let $\hat{\psi}_{SRS}$ and $\hat{\xi}_{SRS}$ be the maximum likelihood estimators of ψ and ξ, respectively, and let $g_0(.;\psi,\xi)$ be the probability distribution function of GQLD and $G_0(.;\psi,\xi)$ be the cumulative distribution function of GQLD. we consider the following test statistics:

- The Kullback-Leibler distance (Kullback and Leibler, 1951) between $g(x)$ and $g_0(x;\psi,\xi)$ is defined as

$$\begin{aligned} KL(g, g_0) &= \int_{-\infty}^{\infty} g(x) \log\left[\frac{g(x)}{g_0(x;\psi,\xi)}\right] dx, \\ &= -H(g) - \int_{-\infty}^{\infty} g(x) \log\left[g_0(x;\psi,\xi)\right]. \end{aligned}$$

Where $H(g)$ is the entropy defined by Shanon (1948) as

$$H(g) = -\int_{-\infty}^{\infty} g(x)\log(g(x))dx,$$

and estimated by Vasicek (1976) by:

$$HV_{tm} = \frac{1}{m}\sum_{i=1}^{m}\log\left[\frac{m}{2t}(x_{(i+t)} - x_{(i-t)})\right],$$

where t is integer less then $m/2$ known as window size and $x_{(i)} = x_{(m)}$ if $i > m$ and $x_{(i)} = x_{(1)}$ if $i < 1$. This estimator converges in probability to $H(g)$ as $m, t \to \infty$ and $\frac{t}{m} \to 0$. Hence, the Kullback-Leibler test is given by Song (2002) by:

$$KL_{tm} = -HV_{tm} - \frac{1}{m}\sum_{i=1}^{m}\log\left[g_0(x_i, \hat{\psi}_{SRS}, \hat{\xi}_{SRS})\right],$$

where the distribution of KL_{tm} is free of ψ and ξ.

- The Kolmogorov-Smirnov test statistics Kolmogorov (1933) and Smirnov (1933):

$$KS = \max\left\{\max_{1\le i\le m}\left[\frac{i}{m} - G_0\ (x_{(i)},\ \hat{\psi}_{SRS}, \hat{\xi}_{SRS})\right], \max_{1\le i\le m}\left[G_0(x_{(i)},\ \hat{\psi}_{SRS}, \hat{\xi}_{SRS}) - \frac{i-1}{m}\right]\right\}.$$

- The Anderson-Darling test statistics Anderson and Darling (1954):

$$A^2 = -2\sum_{i=1}^{m}\left\{\left(i - \frac{1}{2}\right)\log\left[G_0(x_{(i)},\ \hat{\psi}_{SRS}, \hat{\xi}_{SRS})\right] + \left(m - i + \frac{1}{2}\right)\log\left[1 - G_0(x_{(i)},\ \hat{\psi}_{SRS}, \hat{\xi}_{SRS})\right]\right\} - m.$$

- The Cramer-von Mises test statistics Cramér (1928) and von Mises(1928):

$$W^2 = \sum_{i=1}^{m}\left[G_0(x_{(i)}, \hat{\psi}_{SRS}, \hat{\xi}_{SRS}) - \frac{2i-1}{2m}\right]^2 + \frac{1}{12m}.$$

- The Zhang (2002) test statistics:

$$Z_K = \max_{1\le i\le m}\left\{\left(i - \frac{1}{2}\right)\log\left[\frac{i - \frac{1}{2}}{mG_0(x_{(i)}, \hat{\psi}_{SRS}, \hat{\xi}_{SRS})}\right] + \left(m - i + \frac{1}{2}\right)\log\left[\frac{m - i - \frac{1}{2}}{m\left[1 - G_0(x_{(i)}, \hat{\psi}_{SRS}, \hat{\xi}_{SRS})\right]}\right]\right\},$$

$$Z_A = -\sum_{i=1}^{m}\left\{\frac{\log\left[G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]}{m-i+\frac{1}{2}}+\frac{\log\left[1-G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]}{i-\frac{1}{2}}\right\},$$

$$Z_C = \sum_{i=1}^{m}\left[\log\left(\frac{G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})^{-1}-1}{\frac{(m-\frac{1}{2})}{(i-\frac{3}{4})^{-1}}}\right)\right]^2.$$

4 Using RSS

Let $\{X_{[i]j}, i=1,2,\ldots,h; j=1,2,\ldots,r\}$ be a ranked set sample of size $M=hr$ from the GQLD and $z_{(1)}\leq z_{(2)}\leq\ldots\leq z_{(M)}$ its corresponding ordered values and $\hat{\psi}_{RSS}$ and $\hat{\xi}_{RSS}$ be the maximum likelihood estimators of ψ and ξ, respectively, using RSS methods. Thus, above goodness of tests for RSS are:

- The test based on Kullback-Leibler distance is defined as

$$KL_{tm}^{RSS} = -HV_{tm}^{RSS}-\frac{1}{M}\sum_{i=1}^{M}\log\left[g_0(z_i,\hat{\psi}_{RSS},\hat{\xi}_{RSS})\right].$$

- The Kolmogorov-Smirnov test statistics:

$$KS=\max\left\{\max_{1\leq i\leq M}\left[\frac{i}{M}-G_0\ (z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})\right],\max_{1\leq i\leq M}\left[G_0(z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})-\frac{i-1}{M}\right]\right\}.$$

- The Anderson-Darling test statistics:

$$A^2=-2\sum_{i=1}^{M}\left\{\left(i-\frac{1}{2}\right)\log\left[G_0(z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]+\ \left(M-i+\frac{1}{2}\right)\log\left[1-G_0(z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]\right\}-M.$$

- The Cramer-von Mises test statistics:

$$W^2=\sum_{i=1}^{M}\left[G_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})-\frac{2i-1}{2M}\right]^2+\frac{1}{12M}.$$

- The Zhang (2002) test statistics:

$$Z_K=\max_{1\leq i\leq M}\left\{\left(i-\frac{1}{2}\right)\log\left[\frac{i-\frac{1}{2}}{MG_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})}\right]+\left(M-i+\frac{1}{2}\right)\log\left[\frac{M-i-\frac{1}{2}}{M\left[1-G_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]}\right]\right\},$$

$$Z_A = -\sum_{i=1}^{M} \left\{ \frac{\log\left[G_0(z_{(i)}, \hat{\psi}_{RSS}, \hat{\xi}_{RSS})\right]}{M - i + \frac{1}{2}} + \frac{\log\left[1 - G_0(z_{(i)}, \hat{\psi}_{RSS}, \hat{\xi}_{RSS})\right]}{i - \frac{1}{2}} \right\},$$

$$Z_C = \sum_{i=1}^{M} \left[\log \left(\frac{G_0(z_{(i)}, \hat{\psi}_{RSS}, \hat{\xi}_{RSS})^{-1} - 1}{\frac{(M-\frac{1}{2})}{(i-\frac{3}{4})^{-1}}} \right) \right]^2 .$$

where $HV_{tm}^{RSS} = \frac{1}{M} \sum_{i=1}^{M} \log\left[\frac{M}{2t}(z_{(i+t)} - z_{(i-t)})\right]$.

5 Simulation study

In this section, we investigate the performance of the power of the proposed goodness of fit test by using a Monte Carlo study. the study is based on 100,000 samples generated from the GQLD with scale parameter 1 and shape parameter 1 with different sample size using SRS and RSS design. the powers of tests based on KL_{tm} and KL_{tm}^{RSS} depend on the window size t. The problem of choosing the optimal values of t which maximizes the powers subject to m is still open in the field of entropy estimation. Therefore, in our simulations, we have used Grzegorzewski and Wieczorkowski (1999)'s heuristics formula for choosing t as: $t = [\sqrt{m} + 0.5]$.

5.1 Critical values

Tables 1-3 present the critical values of the tests for the GQLD in RSS design for $\gamma = 0.01$, 0.05, 0.1, respectively, We take m from 2 to 9 and the set size h from 2 to 5.

5.2 Power comparison

In order to compare the powers of goodness of fit tests in SRS and RSS designs, we have considered twelve following distributions as alternative distributions:

- the Lindley distribution with parameter 1, Lindley (1).
- the Lindley distribution with parameter 3, Lindley (3).
- the quasi Lindley distribution with scale parameter 1 and shape parameter 1, denoted as Quasi Lindley (1,1).
- the quasi Lindley distribution with scale parameter 1 and shape parameter 2, denoted as Quasi Lindley (1,2).
- the quasi Lindley distribution with scale parameter 3 and shape parameter 1, denoted as Quasi Lindley (3,1).
- the Exponential distribution with mean 1, denoted as Exponential (1).

- the log-logistic distribution with scale parameter 2 and shape parameter 1, denoted as Log-Logistic (2,1).
- the Weibull distribution with scale parameter 1 and shape parameter 2, denoted as Weibull (1,2).
- the Weibull distribution with scale parameter 2 and shape parameter 5, denoted as Weibull (2,5).
- the power Lindley distribution with scale parameter 3 and shape parameter 3, denoted as Power Lindley (3,3).
- the Uniform distribution on (0,1), denoted as Uniform (0,1).
- the generalized Rayleigh distribution with scale parameter 1 and shape parameter 2, denoted as Generalized Rayleigh (1,2).

Tables 4-6 present the estimated powers, respectively, for $M = 10$,20, and 40 for $\gamma = 0.05$ using SRS and RSS methods. In RSS scheme, the value of h (set size) is taken to be 2 and 5, so we can observe the effect of increasing sample size while set size is fixed, and the effect of increasing set size while sample is fixed.

Remarks: Based on simulation results it can be observed that:

- The suggested RSS goodness-of-fit tests are more powerful than their SRS counterparts for all cases considered in this study. As an example considered the case when $M = 10$, $h = 2$ the quasi Lindley distribution with parameters(1,3) as an alternative the power values of the tests KS, A^2, W^2 based on RSS are 0.112, 0.382, 0.132 compared to 0.090, 0.270, 0.099 using RSS, respectively.
- The power of the goodness of fit tests increase as the set size h increase. As illustration, when M=20 for the Exponential distribution observe that $Z_K = 0.799$, $Z_A = 0.827$ and $Z_C = 0.878$ for $h = 2$ and for $h = 5$ $Z_K = 0.969$, $Z_A = 0.978$ and $Z_C = 0.970$.
- The power of the goodness of fit tests increase in the sample size. As an example, based on RSS with $h = 5$ for the Lindley distribution (1), the power values of the Cramer-von Mises test are 0.343, 0.810, 0.998, respectively with $M = 10, 20, 40$.
- In most cases, the large power values are when the alternatives are power Lindley distribution (3,3) and generalized Rayleigh distribution (1,1).
- The power values of the suggested goodness-of-fit tests depend on the distribution parameters for the same test and sample size. As an example when $M = 20$ and $h = 2$ the power of the Anderson-Darling test are 0.777, 0.789, 0.714 for quasi Lindley distribution with parameters (1,1), (1,2), (3,1), respectively.

m	h	KL	KS	A^2	W^2	Z_K	Z_A	Z_C
2	2	2.290	0.573	18.995	0.319	2.017	4.528	13.900
2	3	1.676	0.477	40.109	0.311	2.436	4.278	18.141
2	4	1.339	0.410	69.032	0.297	2.664	4.101	19.710
2	5	1.138	0.359	105.460	0.280	2.770	3.958	20.963
3	2	1.363	0.499	41.284	0.358	2.618	4.379	17.251
3	3	0.982	0.406	88.292	0.340	2.977	4.108	20.630
3	4	0.783	0.347	152.853	0.321	3.093	3.936	22.114
3	5	0.660	0.300	234.643	0.292	3.162	3.810	22.431
4	2	1.099	0.446	71.878	0.382	3.000	4.245	19.698
4	3	0.801	0.361	155.261	0.366	3.369	3.999	22.520
4	4	0.644	0.302	269.224	0.330	3.402	3.830	23.599
4	5	0.546	0.260	412.653	0.288	3.379	3.719	24.108
5	2	0.937	0.408	110.830	0.399	3.277	4.133	21.073
5	3	0.686	0.328	240.371	0.377	3.567	3.906	24.122
5	4	0.555	0.272	417.146	0.334	3.598	3.756	24.814
5	5	0.479	0.234	640.590	0.288	3.548	3.660	25.030
6	2	0.818	0.380	158.162	0.421	3.544	4.061	22.533
6	3	0.609	0.302	343.864	0.390	3.742	3.838	24.843
6	4	0.495	0.248	595.812	0.327	3.744	3.700	25.503
6	5	0.431	0.214	918.307	0.286	3.695	3.614	25.881
7	2	0.695	0.358	213.650	0.435	3.755	3.994	23.693
7	3	0.502	0.281	464.991	0.395	3.852	3.778	25.319
7	4	0.404	0.230	806.901	0.326	3.847	3.654	26.148
7	5	0.342	0.199	1246.402	0.285	3.819	3.580	26.380
8	2	0.626	0.335	277.230	0.446	3.845	3.932	24.501
8	3	0.456	0.265	603.429	0.395	4.026	3.734	26.141
8	4	0.371	0.216	1049.537	0.325	3.967	3.623	26.534
8	5	0.316	0.186	1624.186	0.285	3.885	3.552	26.586
9	2	0.575	0.321	349.609	0.465	4.035	3.890	25.458
9	3	0.421	0.249	759.824	0.392	4.091	3.693	26.713
9	4	0.341	0.204	1325.100	0.325	4.019	3.595	27.388
9	5	0.296	0.175	2051.360	0.281	4.025	3.532	27.425

Table 1: Critical values of different tests of GQLD distribution for different values of (m, h) in RSS design, at significance level $\gamma = 0.01$.

m	h	KL	KS	A^2	W^2	Z_K	Z_A	Z_C
2	2	1.724	0.502	17.197	0.232	1.481	4.143	9.884
2	3	1.287	0.414	37.746	0.223	1.791	3.978	11.618
2	4	1.048	0.353	66.005	0.213	1.933	3.852	12.495
2	5	0.894	0.308	101.846	0.196	1.967	3.748	12.791
3	2	1.058	0.430	38.389	0.250	1.908	4.027	11.981
3	3	0.781	0.350	84.499	0.240	2.169	3.853	13.569
3	4	0.629	0.295	147.862	0.221	2.249	3.736	14.242
3	5	0.531	0.256	227.858	0.197	2.246	3.650	14.361
4	2	0.871	0.385	67.913	0.268	2.211	3.937	13.510
4	3	0.643	0.308	149.530	0.253	2.418	3.772	14.909
4	4	0.521	0.257	260.918	0.220	2.436	3.659	15.251
4	5	0.446	0.222	403.572	0.196	2.450	3.588	15.457
5	2	0.748	0.351	105.643	0.281	2.443	3.873	14.708
5	3	0.554	0.278	232.393	0.256	2.588	3.708	15.944
5	4	0.454	0.231	406.186	0.220	2.590	3.608	16.144
5	5	0.393	0.200	629.326	0.194	2.572	3.545	16.232
6	2	0.661	0.325	151.485	0.291	2.624	3.814	15.668
6	3	0.495	0.256	332.964	0.257	2.713	3.662	16.710
6	4	0.407	0.212	583.191	0.219	2.710	3.570	16.864
6	5	0.356	0.183	905.237	0.194	2.681	3.512	16.700
7	2	0.569	0.304	205.389	0.299	2.740	3.767	16.431
7	3	0.415	0.237	451.138	0.256	2.804	3.624	17.339
7	4	0.334	0.197	792.257	0.217	2.791	3.541	17.392
7	5	0.285	0.169	1230.686	0.193	2.761	3.490	17.368
8	2	0.516	0.287	267.634	0.306	2.860	3.735	17.272
8	3	0.378	0.223	587.463	0.256	2.883	3.591	17.767
8	4	0.306	0.184	1033.301	0.218	2.847	3.515	17.693
8	5	0.264	0.159	1606.635	0.193	2.846	3.471	17.836
9	2	0.475	0.272	337.864	0.314	2.957	3.698	17.817
9	3	0.350	0.211	742.068	0.255	2.965	3.569	18.302
9	4	0.285	0.174	1306.331	0.218	2.926	3.498	18.201
9	5	0.247	0.150	2032.310	0.191	2.897	3.456	18.216

Table 2: Critical values of different tests of GQLD distribution for different values of (m, h) in RSS design, at significance level $\gamma = 0.05$.

m	h	KL	KS	A^2	W^2	Z_K	Z_A	Z_C
2	2	1.468	0.461	16.434	0.190	1.230	3.961	8.245
2	3	1.101	0.380	36.630	0.185	1.489	3.838	9.617
2	4	0.907	0.324	64.615	0.175	1.596	3.736	10.195
2	5	0.786	0.282	100.017	0.158	1.637	3.656	10.440
3	2	0.930	0.396	37.178	0.205	1.596	3.880	10.053
3	3	0.694	0.321	82.716	0.198	1.807	3.745	11.281
3	4	0.558	0.270	145.109	0.176	1.866	3.647	11.715
3	5	0.473	0.233	224.861	0.158	1.869	3.578	11.825
4	2	0.774	0.353	66.162	0.218	1.849	3.811	11.350
4	3	0.572	0.282	146.736	0.202	2.005	3.677	12.418
4	4	0.466	0.235	257.373	0.176	2.031	3.586	12.595
4	5	0.398	0.204	399.668	0.156	2.027	3.526	12.646
5	2	0.665	0.321	103.327	0.229	2.054	3.760	12.409
5	3	0.495	0.254	228.467	0.203	2.163	3.629	13.251
5	4	0.406	0.211	401.713	0.176	2.155	3.544	13.334
5	5	0.353	0.183	624.585	0.157	2.145	3.493	13.326
6	2	0.588	0.297	148.463	0.236	2.180	3.715	13.175
6	3	0.442	0.233	327.895	0.202	2.262	3.589	13.840
6	4	0.367	0.194	577.963	0.175	2.256	3.513	13.839
6	5	0.321	0.167	899.486	0.157	2.231	3.469	13.875
7	2	0.513	0.277	201.697	0.239	2.301	3.675	13.788
7	3	0.374	0.216	445.445	0.200	2.344	3.557	14.273
7	4	0.302	0.179	786.067	0.175	2.330	3.490	14.365
7	5	0.258	0.155	1224.094	0.156	2.308	3.450	14.377
8	2	0.463	0.260	262.593	0.240	2.383	3.643	14.296
8	3	0.340	0.202	580.990	0.200	2.412	3.531	14.674
8	4	0.277	0.168	1026.581	0.175	2.408	3.474	14.807
8	5	0.241	0.146	1599.240	0.157	2.408	3.437	14.894
9	2	0.425	0.246	331.480	0.241	2.456	3.618	14.846
9	3	0.314	0.191	734.457	0.199	2.468	3.512	15.039
9	4	0.258	0.159	1298.817	0.175	2.461	3.457	15.096
9	5	0.224	0.137	2023.869	0.156	2.451	3.425	15.161

Table 3: Critical values of different tests of GQLD distribution for different values of (m, h) in RSS design, at significance level $\gamma = 0.1$.

Sampling scheme	Alternative distribution	Test Statistics						
		KL	KS	A^2	W^2	Z_K	Z_A	Z_C
SRS	Lindley (1)	0.124	0.091	0.286	0.101	0.267	0.291	0.362
	Lindley (3)	0.161	0.111	0.336	0.123	0.327	0.360	0.431
	Quasi Lindley (1,1)	0.125	0.092	0.287	0.100	0.270	0.293	0.363
	Quasi Lindley (1,2)	0.154	0.085	0.323	0.093	0.310	0.349	0.428
	Quasi Lindley (3,1)	0.117	0.090	0.270	0.099	0.258	0.282	0.347
	Exponential (1)	0.186	0.084	0.356	0.092	0.349	0.402	0.491
	Log-Logistic (2,1)	0.134	0.150	0.233	0.179	0.267	0.293	0.399
	Weibul (1,2)	0.183	0.074	0.337	0.080	0.335	0.391	0.479
	Weibul (2,5)	0.090	0.050	0.038	0.029	0.074	0.085	0.095
	Power Lindley (3,3)	0.814	0.907	0.982	0.980	0.873	0.921	0.915
	Uniform (0,1)	0.340	0.407	0.615	0.482	0.456	0.329	0.346
	Generalized Rayleigh (1,2)	0.759	0.880	0.970	0.969	0.846	0.930	0.911
RSS (h =2)	Lindley (1)	0.164	0.124	0.422	0.148	0.378	0.387	0.452
	Lindley (3)	0.190	0.113	0.411	0.133	0.391	0.420	0.480
	Quasi Lindley (1,1)	0.163	0.125	0.420	0.148	0.378	0.387	0.451
	Quasi Lindley (1,2)	0.207	0.111	0.457	0.136	0.426	0.450	0.520
	Quasi Lindley (3,1)	0.153	0.112	0.382	0.132	0.346	0.359	0.420
	Exponential (1)	0.253	0.105	0.494	0.132	0.478	0.516	0.590
	Log-Logistic (2,1)	0.181	0.161	0.310	0.198	0.384	0.404	0.489
	Weibul (1,2)	0.247	0.097	0.470	0.120	0.460	0.501	0.575
	Weibul (2,5)	0.117	0.135	0.124	0.124	0.147	0.195	0.165
	Power Lindley (3,3)	0.802	0.934	0.994	0.994	0.865	0.910	0.896
	Uniform (0,1)	0.327	0.481	0.725	0.590	0.481	0.312	0.306
	Generalized Rayleigh (1,2)	0.742	0.910	0.989	0.988	0.838	0.923	0.895
RSS (h =5)	Lindley (1)	0.266	0.266	0.840	0.343	0.631	0.631	0.600
	Lindley (3)	0.271	0.167	0.625	0.222	0.553	0.572	0.569
	Quasi Lindley (1,1)	0.267	0.266	0.838	0.346	0.633	0.628	0.600
	Quasi Lindley (1,2)	0.340	0.241	0.856	0.322	0.687	0.702	0.676
	Quasi Lindley (3,1)	0.240	0.232	0.742	0.300	0.572	0.577	0.555
	Exponential (1)	0.416	0.222	0.873	0.304	0.739	0.762	0.742
	Log-Logistic (2,1)	0.296	0.223	0.761	0.295	0.594	0.626	0.615
	Weibul (1,2)	0.411	0.213	0.854	0.294	0.728	0.751	0.732
	Weibul (2,5)	0.188	0.394	0.746	0.660	0.340	0.458	0.339
	Power Lindley (3,3)	0.864	0.991	1.000	1.000	0.942	0.954	0.933
	Uniform (0,1)	0.361	0.722	0.952	0.878	0.636	0.372	0.304
	Generalized Rayleigh (1,2)	0.799	0.983	1.000	1.000	0.923	0.961	0.933

Table 4: Power estimates of different goodness of fit tests in SRS and RSS designs for $M = 10$ and γ=0.05.

Sampling scheme	Alternative distribution	Test Statistics						
		KL	KS	A^2	W^2	Z_K	Z_A	Z_C
SRS	Lindley (1)	0.341	0.136	0.478	0.143	0.524	0.510	0.656
	Lindley (3)	0.398	0.126	0.465	0.129	0.536	0.561	0.680
	Quasi Lindley (1,1)	0.341	0.135	0.476	0.142	0.523	0.513	0.659
	Quasi Lindley (1,2)	0.427	0.120	0.517	0.125	0.585	0.602	0.741
	Quasi Lindley (3,1)	0.322	0.127	0.441	0.132	0.490	0.480	0.620
	Exponential (1)	0.505	0.114	0.551	0.122	0.646	0.685	0.814
	Log-Logistic (2,1)	0.301	0.162	0.287	0.187	0.525	0.550	0.758
	Weibul (1,2)	0.493	0.105	0.524	0.112	0.622	0.667	0.792
	Weibul (2,5)	0.187	0.166	0.226	0.169	0.234	0.417	0.348
	Power Lindley (3,3)	0.990	0.998	1.000	1.000	0.997	0.999	0.999
	Uniform (0,1)	0.812	0.758	0.932	0.821	0.839	0.611	0.707
	Generalized Rayleigh (1,2)	0.973	0.996	1.000	1.000	0.995	0.999	0.999
RSS (h =2)	Lindley (1)	0.428	0.254	0.777	0.301	0.697	0.696	0.768
	Lindley (3)	0.462	0.179	0.629	0.211	0.640	0.662	0.730
	Quasi Lindley (1,1)	0.426	0.254	0.777	0.303	0.696	0.695	0.767
	Quasi Lindley (1,2)	0.525	0.224	0.789	0.271	0.754	0.769	0.829
	Quasi Lindley (3,1)	0.404	0.230	0.714	0.270	0.653	0.652	0.726
	Exponential (1)	0.606	0.202	0.796	0.251	0.799	0.827	0.878
	Log-Logistic (2,1)	0.404	0.201	0.524	0.248	0.686	0.758	0.836
	Weibul (1,2)	0.599	0.191	0.773	0.240	0.785	0.815	0.868
	Weibul (2,5)	0.260	0.414	0.863	0.706	0.429	0.676	0.596
	Power Lindley (3,3)	0.990	1.000	1.000	1.000	0.999	0.999	0.999
	Uniform (0,1)	0.827	0.881	0.978	0.941	0.893	0.650	0.712
	Generalized Rayleigh (1,2)	0.971	0.999	1.000	1.000	0.997	1.000	1.000
RSS (h =5)	Lindley (1)	0.616	0.636	1.000	0.810	0.935	0.943	0.921
	Lindley (3)	0.580	0.372	0.854	0.502	0.800	0.812	0.819
	Quasi Lindley (1,1)	0.615	0.639	1.000	0.811	0.936	0.943	0.922
	Quasi Lindley (1,2)	0.722	0.599	1.000	0.765	0.956	0.965	0.951
	Quasi Lindley (3,1)	0.585	0.593	0.976	0.758	0.903	0.909	0.890
	Exponential (1)	0.802	0.552	1.000	0.721	0.969	0.978	0.970
	Log-Logistic (2,1)	0.567	0.383	0.995	0.534	0.899	0.947	0.935
	Weibul (1,2)	0.800	0.549	0.998	0.714	0.968	0.976	0.968
	Weibul (2,5)	0.408	0.881	0.998	0.994	0.804	0.966	0.933
	Power Lindley (3,3)	0.996	1.000	1.000	1.000	1.000	1.000	1.000
	Uniform (0,1)	0.911	0.991	0.999	0.998	0.984	0.887	0.879
	Generalized Rayleigh (1,2)	0.983	1.000	1.000	1.000	1.000	1.000	1.000

Table 5: Power estimates of different goodness of fit tests in SRS and RSS designs for $M = 20$ and γ=0.05.

Sampling scheme	Alternative distribution	Test Statistics						
		KL	KS	A^2	W^2	Z_K	Z_A	Z_C
SRS	Lindley (1)	0.750	0.355	0.927	0.401	0.932	0.997	1.000
	Lindley (3)	0.775	0.262	0.818	0.300	0.869	0.908	0.937
	Quasi Lindley (1,1)	0.750	0.356	0.926	0.400	0.932	0.997	1.000
	Quasi Lindley (1,2)	0.835	0.309	0.931	0.359	0.955	0.999	1.000
	Quasi Lindley (3,1)	0.734	0.338	0.897	0.380	0.911	0.972	0.983
	Exponential (1)	0.895	0.282	0.931	0.344	0.970	0.999	1.000
	Log-Logistic (2,1)	0.630	0.202	0.503	0.239	0.900	0.999	1.000
	Weibul (1,2)	0.886	0.272	0.917	0.331	0.959	0.990	0.994
	Weibul (2,5)	0.537	0.645	0.963	0.918	0.763	0.965	0.968
	Power Lindley (3,3)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Uniform (0,1)	0.997	0.988	0.997	0.992	0.996	0.990	0.993
	Generalized Rayleigh (1,2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
RSS (h =2)	Lindley (1)	0.840	0.668	0.999	0.779	0.985	0.998	0.999
	Lindley (3)	0.815	0.447	0.907	0.551	0.908	0.926	0.945
	Quasi Lindley (1,1)	0.840	0.670	0.999	0.779	0.984	0.998	0.999
	Quasi Lindley (1,2)	0.905	0.615	0.999	0.729	0.990	0.999	0.999
	Quasi Lindley (3,1)	0.824	0.643	0.987	0.751	0.971	0.985	0.989
	Exponential (1)	0.944	0.556	0.998	0.683	0.994	0.999	1.000
	Log-Logistic (2,1)	0.761	0.298	0.903	0.380	0.970	0.999	1.000
	Weibul(1,2)	0.943	0.547	0.997	0.673	0.993	0.998	0.999
	Weibul(2,5)	0.693	0.940	0.998	0.993	0.951	0.995	0.995
	Power Lindley (3,3)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Uniform (0,1)	0.999	0.999	1.000	0.999	0.999	0.998	0.999
	Generalized Rayleigh (1,2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
RSS (h =5)	Lindley (1)	0.948	0.980	1.000	0.998	1.000	1.000	1.000
	Lindley (3)	0.883	0.713	0.979	0.838	0.962	0.968	0.971
	Quasi Lindley (1,1)	0.949	0.979	1.000	0.998	1.000	1.000	1.000
	Quasi Lindley (1,2)	0.978	0.970	1.000	0.995	1.000	1.000	1.000
	Quasi Lindley (3,1)	0.940	0.968	1.000	0.993	0.998	0.999	0.999
	Exponential (1)	0.990	0.955	1.000	0.990	1.000	1.000	1.000
	Log-Logistic (2,1)	0.888	0.640	1.000	0.768	0.999	1.000	1.000
	Weibul(1,2)	0.991	0.953	1.000	0.989	1.000	1.000	1.000
	Weibul(2,5)	0.839	1.000	1.000	1.000	1.000	1.000	1.000
	Power Lindley (3,3)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Uniform (0,1)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Generalized Rayleigh (1,2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 6: Power estimates of different goodness of fit tests in SRS and RSS designs for $M = 40$ and γ=0.05.

6 An application

In this section, we show the useful of our proposed RSS-goodness of fit by a real data example which present the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data points are

Data set: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

We present in Table 7 Akaike information criterion (AIC) introduced by Akaike (1974), Baysian information criterion (BIC) proposed by Schwarz (1978), HannanQuinn Information Criterion (HQIC) suggested by Hannan and Quinn (1979), Consistent Akaike Information Criterion (CAIC) by Bozdogan (1987), KS distance and its corresponding p-value.

Table 7: AIC, AICc, BIC, HQIC,K-S distance and p-value for data set.

AIC	AICc	BIC	HQIC	K-S	p-value
209.5955	209.7694	214.1488	211.4082	0.092806	0.564624

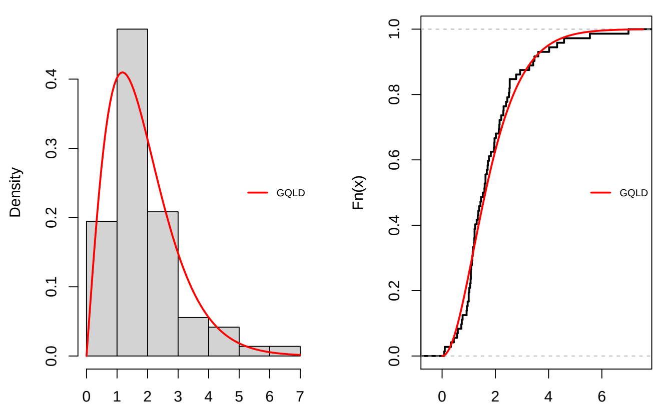


Figure 2: Plots of estimated probability density functions and cumulative distribution functions for data set.

It is clear from Table 7 and Figure 2 that the GQLD present a good fit to the data set. Hence, we apply our proposed goodness of fit on a ranked set sample from the data of size $M = 10$ and set size $h = 5$. The sampled values are:

$$0.72, 1.63, 2.22, 1.71, 5.55, 0.10, 0.77, 1.53, 1.72, 4.32.$$

The estimated parameters using maximum likelihood method are $\hat{\psi} = 1.259$ and $\hat{\xi} = 0.671$, the values of all the test statistics are computed and given in Table 8.

KL	KS	A^2	W^2	Z_K	Z_A	Z_C
0.452	0.358	106.363	0.276	1.966	3.824	15.719

Table 8: Computing values of different test statistics in RSS.

By comparing these values with the corresponding critical values in Table 2, we observe that the null hypothesis that the data follow a GQLD is not rejected by KL and Z_K at significance level of $\gamma = 0.05$.

7 Conclusion

In this paper, we developed goodness of fit test for the GQLD using SRS and RSS methods. The forms of the tests are presented. A simulation study is conducted to evaluate the usefulness of the suggested goodness of fit tests. An application to areal data set is presented for illustration. It is found that tests based on RSS are more powerful than their competitors based on SRS method. As a future works, the author recommend to investigate the suggested tests based on other RSS methods.

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