FRUSTRATING SOLUTIONS FOR TWO EXPONENTIAL DIOPHANTINE EQUATIONS $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$ AND $(\mathbb{p}+1)^a - \mathbb{p}^b + 1 = c^2$

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Abstract:

In this paper, two dissimilar exponential Diophantine equations $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$ and $(\mathbb{p}+1)^a - \mathbb{p}^b + 1 = c^2$ where $\mathbb{p} = 2^p - 1$ is a Mersenne prime for any prime $p > 2$ and $a, b, c \in Z^+$ are deliberated and proved that $(0, 2n, (\mathbb{p}+3)^n), (2n, 0, \mathbb{p}^n)$ and $\left(1,1,2^{\frac{p+1}{2}}\right)$, $n \in \mathbb{N}$ signifies solutions to the first equation and $(2n, 0, (\mathbb{p}+1)^n)$ and $(2,2,2^{\frac{p+1}{2}})$ embodies solutions to the later equation respectively.

Keywords:

Congruence, Mersenne prime, Exponential Diophantine equation.

I.Introduction:

A Diophantine problem is one in which the solutions are mandatory to be integers. If a Diophantine equation has a supplementary variable or variables occurring as exponents, it is known as an exponential Diophantine equation [1-4].In [5-7], the authors discussed an exponential Diophantine equation $p^x + (p+6)^y = z^2$ where p is any prime number for finding integer solutions. For a wide-ranging analysis one can refer [8-10]. In this communication, two peculiar exponential Diophantine equations $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$ and $(\mathbb{p}+1)^a - \mathbb{p}^b + 1 = c^2$ where $\mathbb{p} = (2^P - 1)$ for any prime $p > 2$ is a Mersenne prime and a, b stands for non-negative integers are exasperating for verdict integer solutions. Also, the equation $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$ is evidenced that has infinitely many solutions $(a,b,c) = \left\{(0,2n,(\mathbb{p}+3)^n), (2n,0,\mathbb{p}^n), (1,1,2^{\frac{p+1}{2}})\right\}$ where $n \in \mathbb{N}$ and the solutions to $(\mathbb{p}+1)^a - (\mathbb{p})^b + 1 = c^2$ are $(a,b,c) = \left\{(2n,0,(\mathbb{p}+1)^n), (2,2,2^{\frac{p+1}{2}})\right\}$.

II.Frustrating solutions for an exponential Diophantine Equation $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$

In this section, the existence of solutions of the Diophantine equation $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$ is analysed by using the elementary concepts of congruence as follows.

Lemma 1:

The Diophantine equation $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$ where $\mathbb{p} = 2^p - 1$ is a Mersenne prime for any prime $p > 2$ has solutions $(a, b, c) = \{(0, 2n, (\mathbb{p}+3)^n), (2n, 0, \mathbb{p}^n)\}$, $n \in \mathbb{N}$.

Proof:

Since any square integer is congruence to either $0(mod\ 4)$ or $1(mod\ 4)$,

$$c^2 \equiv 0(mod\ 4) \text{ or } c^2 \equiv 1(mod\ 4)$$

Case 1: If $a = 0$, then the desired equation becomes $(\mathbb{p}+3)^b = c^2$. For this case, let us consider the following two sub cases.

Sub case i: If b is even, then $b = 2n, n \in \mathbb{N}$. Thus, $c = (\mathbb{p}+3)^n$. Hence the solutions to the equation are monitored by $(0, 2n, (\mathbb{p}+3)^n)$.

Sub case ii: If b is odd, then $b = 2n-1, n \in \mathbb{N}$. Thus $c^2 = (\mathbb{p}+3)^{2n-1} = \frac{(\mathbb{p}+3)^{2n}}{(\mathbb{p}+3)}$. For any Mersenne prime $\mathbb{p}$, it is noted that $\mathbb{p}+3$ can never be a square. Therefore, c is not an integer. Hence, this case does not possess an integer solution.

Case 2: If $b = 0$, then the chosen equation becomes $\mathbb{p}^a = c^2$. Here, the following two sub cases possible.

Sub case i: If a is even, then $a = 2n, n \in \mathbb{N}$. Thus, $c = \mathbb{p}$.Hence, the generalized solutions to the equation are given by $(2n, 0, \mathbb{p}^n), n \in \mathbb{N}$.

Sub case ii: If a is odd, then $a = 2n-1, n \in \mathbb{N}$. Thus $c^2 = \mathbb{p}^{2n-1} = \frac{\mathbb{p}^{2n}}{\mathbb{p}}$. Since any Mersenne prime $\mathbb{p}$ can never be a square, c cannot be an integer. Hence, for this case there exists no solutions in integers to the equation.

Theorem 1:

Let $\mathbb{p} = 2^p - 1$ is a Mersenne prime for any prime $p > 2$. If $a.b = 0,1,2,3,4$ then the Diophantine equation $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$ has infinitely many integer solutions $(a,b,c) = \left\{(0,2n,(\mathbb{p}+3)^n),(2n,0,\mathbb{p}^n),\left(1,1,2^{\frac{p+1}{2}}\right)\right\}, n \in \mathbb{N}$.

Proof:

Let a, b, c be non-negative integers and $\mathbb{p}$ satisfies our assumption. Then there are the following five cases. It is well known that

$$c^2 \equiv 0(mod\ 4) \text{ or } c^2 \equiv 1(mod\ 4).$$

Case 1:$a.b = 0$

Sub case i: Suppose $a = 0, b = 1$. Then the equation reduces to $\mathbb{p} + 3 = c^2$ which is impossible for any Mersenne prime. Hence there cannot occurs solutions in integer.

Sub case ii: Suppose $a = 0, b > 1$. By Lemma 1, the infinitely many solutions to the equation are obtained as $(0,2n,(\mathbb{p}+3)^n), n \in \mathbb{N}$.

Sub case iii: Suppose $a = 1, b = 0$. Then $\mathbb{p} = c^2$ which is absurd. Hence, in this case there exists no solution to the peculiar equation.

Sub case iv: Suppose $a > 1, b = 0$. By Lemma 1, the sequence of integer solutions is given by $(2n, 0, \mathbb{p}^n), n \in \mathbb{N}$.

Case 2: $a.b = 1$.The only options for a and b are $a = 1, b = 1$. Then the equation can be written as $2\mathbb{p} + 2 = c^2 \Rightarrow 2^{\mathbb{p}+1} = c^2$.Then $c = 2^{\frac{p+1}{2}}$ where $p > 2$. Therefore, the triple $\left(1,1,2^{\frac{p+1}{2}}\right)$ is a solution to the equation.

Case 3: $a.b = 2$

Sub case i: Suppose $a = 1, b = 2$. Then, the original equation is simplified to

$\mathbb{p} + (\mathbb{p}+3)^2 - 1 = c^2$. Since $\mathbb{p} \equiv 3\ (mod\ 4)$ and $(\mathbb{p}+3)^2 \equiv 0\ (mod\ 4)$,

$$\mathbb{p} + (\mathbb{p}+3)^2 - 1 \equiv 2(mod\ 4)$$

Sub case ii: Suppose $a = 2, b = 1$. Then, the considered equation be reduced to

$\mathbb{p}^2 + \mathbb{p} + 2 = c^2$. since $\mathbb{p} \equiv 3(mod\ 4)$ and $\mathbb{p}^2 \equiv 1\ (mod\ 4)$,

$$\mathbb{p}^2 + \mathbb{p} + 2 \equiv 2(mod\ 4)$$

Case 4: $a.b = 3$

Sub case i: Take $a = 1, b = 3$. Then the corresponding form of an inventive equation is $\mathbb{p} + (\mathbb{p} + 3)^3 - 1 = c^2$. But $\mathbb{p} \equiv 3\ (mod\ 4)$ and $(\mathbb{p} + 3)^3 \equiv 0\ (mod\ 4)$ implies that

$$\mathbb{p} + (\mathbb{p} + 3)^3 - 1 \equiv 2(mod\ 4)$$

Sub case ii: Assume that $a = 3, b = 1$. Then the equation becomes $\mathbb{p}^3 + (\mathbb{p} + 3) - 1 \equiv c^2$.

Now, $\mathbb{p}^3 + (\mathbb{p} + 3) - 1 = (2^p - 1)^3 + (2^p + 1) = 2^P[2^{2p} - 3.2^p + 4]$

which can never be a square for any prime $p > 2$. Hence this case does not possess an integer solution.

Case 5: $a.b = 4$

Sub case i: Choose $a = 1, b = 4$. Then, the equation can be modified into $\mathbb{p} + (\mathbb{p} + 3)^4 - 1 = c^2$. Since $\mathbb{p} \equiv 3\ (mod\ 4)$ and $(\mathbb{p} + 3)^4 \equiv 0\ (mod\ 4)$.

$$\mathbb{p} + (\mathbb{p} + 3)^4 - 1 \equiv 2(mod\ 4)$$

Sub case ii: Assume that $a = 4, b = 1$.These choices reduces the unique equation to $\mathbb{p}^4 + \mathbb{p} + 2 = c^2$. Then $\mathbb{p} \equiv 3(mod\ 4)$ and $\mathbb{p}^4 \equiv 1\ (mod\ 4)$ produces that

$$\mathbb{p}^4 + \mathbb{p} + 2 \equiv 2(mod\ 4)$$

Sub case iii: Take $a = 2, b = 2$. Then, the initial equation can be rewritten as $2\mathbb{p}^2 + 6\mathbb{p} + 9 = c^2$ which cannot be a true for any Mersenne prime.

Thus, for all choices of a and b in case 3,case 4 and case 5 the left-hand side of the expression is congruent to 2 (mod 4) whereas the right-hand side is congruent to either $0\ (mod\ 4)$ or $1(mod\ 4)$.

Subsequently, for these three cases, there cannot occurs an integer solution. Hence, it is concluded that the generalized sequences of integer solution to the equation are $(a,b,c) = \left\{(0,2n,(\mathbb{p}+3)^n),(2n,0,\mathbb{p}^n),\left(1,1,2^{\frac{p+1}{2}}\right)\right\}, n \in \mathbb{N}$.

Corollary 1:

The Diophantine equation $(\mathbb{p}-3)^a + \mathbb{p}^b - 1 = c^2$ has infinitely many integer solutions

$(a,b,c) = \{(2n,0,\mathbb{p}^n),(0,2n,(\mathbb{p}-3)^n)\}, n \in \mathbb{N}$.

Corollary 2:

The Diophantine equation $(\mathbb{p}+3)^a + \mathbb{p}^b - 1 = c^2$ where $\mathbb{p} = (2^p - 1)$ for any prime $p > 2$ has the solutions$(a,b,c) = \left\{(0,2n,\mathbb{p}^n),(2n,0,(\mathbb{p}+3)^n),(1,1,2^{\frac{p+1}{2}})\right\}$ where $n \in \mathbb{N}$

Corollary 3:

The solution of the Diophantine equation $(\mathbb{p}+3)^a + \mathbb{p}^b - 1 = c^{2u}, u \geq 1$ is $(1,1,\ 2^{\frac{p+1}{2u}})$.

The consolidated results for the specified equation are tabularized below:

a	b	Reduced form of $\mathbb{p}^a + (\mathbb{p}+3)^b - 1 = c^2$	Analysis of solution
1	0	$\mathbb{p} = c^2$	No solution
0	1	$\mathbb{p} + 3 = c^2$	No solution
1	1	$\mathbb{p} + (\mathbb{p}+3) - 1 = c^2$	$(1,1,\ 2^{\frac{p+1}{2}})$
$2n$	0	$\mathbb{p}^{2n} = c^2$	$(2n,0,\mathbb{p}^n)$
0	$2n$	$(\mathbb{p}+3)^{2n} = c^2$	$(0,2n,(\mathbb{p}+3)^n)$

III. Frustrating solutions for an exponential Diophantine Equation $(\mathbb{p}+1)^a - \mathbb{p}^b + 1 = c^2$

In this section, the existence of the Diophantine equation $(\mathbb{p}+1)^a - \mathbb{p}^b + 1 = c^2$ is discussed by using the basic concepts of congruence as follows.

Lemma 2:

The Diophantine equation $(\mathbb{p}+1)^a - \mathbb{p}^b + 1 = c^2$ where $\mathbb{p} = 2^p - 1$ is a Mersenne prime for any prime $p > 2$ has a solution $(a,b,c) = (2n,0,(\mathbb{p}+1)^n), n \in \mathbb{N}$.

Proof:

The paradox is $c^2 \equiv 0(mod\ 4)$ or $c^2 \equiv 1(mod\ 4)$.

Case 1: If $a = 0$, then the preferred equation becomes $2 - \mathbb{p}^b = c^2$.Let us pick the next two sub cases.

Sub case i: If $b = 1$, then the consequent equation is $2 - \mathbb{p} = c^2 \Rightarrow -2^p + 3 = c^2$

which is not possible for any prime $p > 2$.

Sub case ii: If $b > 1$, then the required equation is $\mathbb{p}^b = 2 - c^2, n \in \mathbb{N}$.

Since,$c^2 \equiv \begin{cases} 0(mod\ 4)\ if\ c\ is\ even \\ 1(mod\ 4)\ if\ c\ is\ odd \end{cases}$

$$2 - c^2 \equiv \begin{cases} 2(mod\ 4)\ \ if\ \ c\ is\ even \\ 1(mod\ 4)\ if\ c\ is\ odd \end{cases}$$

But $\mathbb{p}^b \equiv \begin{cases} 1(mod\ 4)\ if\ c\ is\ even \\ 3(mod\ 4)\ if\ c\ is\ odd \end{cases}$

This contradiction shows that the equation has no solution in integer.

Case 2: If $b = 0$, then the needed equation is converted into $(\mathbb{p} + 1)^a = c^2$. Contemplate the following two sub cases.

Sub case i: If a is even, then $a = 2n, n \in \mathbb{N}$. Thus, $c = \mathbb{p} + 1$.Hence, the solutions to the equation are observed by $(2n, 0, (\mathbb{p} + 1)^n), n \in \mathbb{N}$.

Sub case ii: If a is odd, then $a = 2n - 1, n \in \mathbb{N}$. Thus $c^2 = (\mathbb{p} + 1)^{2n-1} = \frac{(\mathbb{p}+1)^{2n}}{\mathbb{p}+1}$

Here, $\mathbb{p} + 1$ is not a square for any Mersenne prime $\mathbb{p}$. Therefore, c is not an integer. Consequently, there exists no integer solution.

Theorem 2:

Let $\mathbb{p} = 2^p - 1$ is a Mersenne prime for any prime $p > 2$. If $a.b = 0,1,2,3,4$, then the Diophantine equation $(\mathbb{p} + 1)^a - \mathbb{p}^b + 1 = c^2$ has enormously many integer solutions $(2n, 0, (\mathbb{p} + 1)^n)$ where $n \in \mathbb{N}$, and $\left(2,2,2^{\frac{p+1}{2}}\right)$.

Proof:

Let us prove the theorem by considering the following five cases ifthe variables a, b, c to be evaluated are positive integers

Case 1:$a.b = 0$

Sub case i: Suppose $a = 0, b \geq 1$.Then the equation condenses to $2 - \mathbb{p}^b = c^2$. By lemma 2, it is verified that the implicit equation has no integer solution.

Sub case ii: Suppose $a = 1, b = 0$. These choices of a and b lead the presumed equation into $\mathbb{p} + 1 = c^2$ which is impossible. Hence, it is not possible to find an integer solution.

Sub case iii: Suppose $a > 1, b = 0$.By Lemma 2, the sequence of integer solutions is invented by $(2n, 0, (\mathbb{p} + 1)^n), n \in \mathbb{N}$.

Case 2: $a.b = 1$.The only possibilities of a and b is $a = 1, b = 1$.The corresponding equation reduces to $2 = c^2$ which is absurd. Consequently, the equation has no integer solutions

Case 3:$a.b = 2$

Sub case i: Suppose $a = 2, b = 1$. For this case the resulting equation is $(\mathbb{p} + 1)^2 - \mathbb{p} + 1 = c^2$.

Since $(\mathbb{p} + 1)^2 \equiv 0(mod\ 4)$ and $\mathbb{p} \equiv 3(mod\ 4), (\mathbb{p} + 1)^2 - \mathbb{p} + 1 \equiv 2(mod\ 4)$.

Sub case ii: Suppose $a = 1, b = 2$.The equation to be scrutinized is $\mathbb{p} - \mathbb{p}^2 + 2 = c^2$. Now,$\mathbb{p} - \mathbb{p}^2 + 2 = 3.2^{\mathbb{p}} - 2^{2p}$ which can never be a square for any prime $p > 2$. So, the equation has no solutions in integer.

Case 4:$a.b = 3$

Sub case i: Take $a = 1, b = 3$. Then the equivalent form of the equation is

$\mathbb{p} - \mathbb{p}^3 + 2 = c^2$. Here $\mathbb{p} - \mathbb{p}^3 + 2 \equiv 2(mod\ 4)$.

Sub case ii: Suppose $a = 3, b = 1$.Then the corresponding form of the equation is $(\mathbb{p} + 1)^3 - \mathbb{p} + 1 = c^2$.

Since $\mathbb{p} \equiv 3(mod\ 4)$ and $(\mathbb{p} + 1)^3 \equiv 0(mod\ 4), (\mathbb{p} + 1)^3 - \mathbb{p} + 1 \equiv 2(mod\ 4)$.

Case 5:$a.b = 4$

Sub case i: Take $a = 1, b = 4$. Then,the equation can be altered into $\mathbb{p} - \mathbb{p}^4 + 2 = c^2$

Now, $\mathbb{p} - \mathbb{p}^4 + 2 = -2^{4p} + 4 \times 2^{3p} - 6 \times 2^{2p} + 5 \times 2^p$which cannot not a square for any prime $p > 2$.

Sub case ii: Choose $a = 4, b = 1$.Then $(\mathbb{p} + 1)^4 - \mathbb{p} + 1 = c^2$

Now, $\mathbb{p} \equiv 3(mod\ 4)$ and $(\mathbb{p} + 1)^4 \equiv 0\ (mod\ 4)$ together implies that

$$(\mathbb{p} + 1)^4 - \mathbb{p} + 1 \equiv 2(mod\ 4)$$

Thus, for all choices of a and b in first subcase of case 3, case 4 and first two subcases of case 5 the left-hand side of the expression is congruent to 2 (mod 4).However, the right-hand side of the expression is congruent to either $0\ (mod\ 4)$ or $1(mod\ 4)$.
Hence, the principal equation cannot have an integer solution.

Sub case iii: Suppose $a = 2, b = 2$.Then the primary equation can be rewritten as

$$2\mathbb{p} + 2 = c^2 \Rightarrow 2^{p+1} = c^2$$

$$\Rightarrow c = 2^{\frac{p+1}{2}}$$

Hence the solutions for the equation are $\left(2,2,2^{\frac{p+1}{2}}\right)$.

The conclusion of the theorem is the complete integer solutions to the equation are $(2n, 0, (\mathbb{p} + 1)^n)$ where $n \in \mathbb{N}$ and $(2,2,2^{\frac{p+1}{2}})$.

Corollary 4:

The Diophantine equation $(\mathbb{p} - 1)^a - \mathbb{p}^b + 1 = c^2$ has infinitely many integer solutions $(2n, 0, (\mathbb{p} - 1)^n), n \in \mathbb{N}$.

Corollary 5:

The solution to the Diophantine equation $(\mathbb{p} + 1)^b - (\mathbb{p})^a - 1 = c^2$ are $(0,2n, (\mathbb{p} + 1)^n)$ where $n \in \mathbb{N}$ and $\left(2,2,2^{\frac{p+1}{2}}\right)$.

Corollary 6:

The Diophantine equation $(\mathbb{p}+1)^a-\mathbb{p}^b+1=c^{2u}$ where $\mathbb{p}=2^P-1$ for any prime $P>2$ has $\left(2,2,2^{\frac{p+1}{2u}}\right)$ as solution.

The detailed results for the definite equation are tabulated below:

a	b	Reduced form of $(\mathbb{p}+1)^a-\mathbb{p}^b+1=c^2$	Analysis of solution
1	0	$\mathbb{p}+1=c^2$	No solution
0	1	$\mathbb{p}=2-c^2$	No solution
1	1	$2=c^2$	No solution
2	2	$(\mathbb{p}+1)^2-\mathbb{p}^2+1=c^2$	$(2,2,2^{\frac{p+1}{2}})$
$2n$	0	$(\mathbb{p}+1)^{2n}=c^2$	$(2n,0,(\mathbb{p}+1)^n)$
0	n	$\mathbb{p}^n=2-c^2$	No solution

IV.Conclusion:

In this paper, two different exponential Diophantine equations $\mathbb{p}^a+(\mathbb{p}+3)^b-1=c^2$ and$(\mathbb{p}+1)^a-\mathbb{p}^b+1=c^2$ where $\mathbb{p}=2^p-1$ is a Mersenne prime for any prime $p>2$ and $a,b,c\in Z^+$ are considered and shown that $(0,2n,(\mathbb{p}+3)^n),(2n,0,\mathbb{p}^n)$ and $\left(1,1,2^{\frac{p+1}{2}}\right)$, $n\in\mathbb{N}$ indicates solutions to the first equation and $(2n,0,(\mathbb{p}+1)^n)$ and $(2,2,2^{\frac{P+1}{2}})$ exemplifies solutions to the second equation respectively. In this method, one can study numerous varieties of exponential Diophantine equations comprising some other prime numbers and can analyse the existence of solutions.

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