INVENTION OF FOUR NOVEL SEQUENCES AND THEIR PROPERTIES

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Abstract- In this paper, four novel sequences named as Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are discovered. Also, the recurrence relations, the general formulae for all sequences and some theorems are invented by exploiting basic concepts of matrices.

Keywords- Pan-San sequence, Pan-San Buddy sequence, Pan-San Comrade sequence and Pan-San Mate sequence, characteristic equation, eigenvalues.

1. INTRODUCTION

Assume that $d \neq 1$ is any positive square free integer. Then the equation $x^2 - dy^2 = 1$ is called as the classical Pell equation. There are numerous integer solutions (x_n, y_n) for $n \geq 0$, for this Pell equation. Many authors such as Lenstra [4], Matthews [5], Techan [9] and others take some certain Pell equations and solutions into account. Pandichelvi .V, Sandhya .P [7] discovered two tremendous sequences Cheldhiya and Cheldhiya companion sequences by taking $d = k^2 + 1$ in the Pell equation $x^2 - dy^2 = \pm 1$. The Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are formed in this communication for the solution to the equation $x^2 - dy^2 = 1$ for some specific d. A few theorems are also supported using these sequences.

2. PAN-SAN AND PAN-SAN BUDDY SEQUENCES

The C and D values in the universal equation $D^2 - dC^2 = 1$ for certain non-zero square-free integer $d = k^2 + 2, k \in N - \{1\}$ propagates two novel sequences $0, k, 2k(k^2+1), 4k(k^2+1)^2 - k, 8k^2(k^2+1)^3 - 4k(k^2+1)^2, \ldots$ and $1, k^2+1, 2(k^2+1)^2 - 1, 4(k^2+1)^3 - 3\,(k^2+1), \ldots$ and named as Pan-San sequence and Pan-San Buddy sequence respectively. The nth term of the first sequence is interpreted by the recurrence relation

$$C_{n,k} = 2(k^2+1)C_{n-1,k} - C_{n-2,k}, \qquad k, n \in N - \{1\}$$

where $C_{0,k} = 0, C_{1,k} = k$.

The nth term of the second sequence is standardized by the recurrence relation

$$D_{n,k} = 2(k^2+1)D_{n-1,k} - D_{n-2,k}, \qquad k, n \in N - \{1\}$$

where $D_{0,k} = 1, D_{1,k} = k^2 + 1$.

Define the Pan-San sequence matrix as

$$\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathcal{M}\begin{pmatrix} C_{1,k} \\ C_{0,k} \end{pmatrix} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} 2k(k^2+1) \\ k \end{pmatrix} = \begin{pmatrix} C_{2,k} \\ C_{1,k} \end{pmatrix}$$

Also,

$$\mathcal{M}\begin{pmatrix} C_{2,k} \\ C_{1,k} \end{pmatrix} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2k(k^2+1) \\ k \end{pmatrix} = \begin{pmatrix} 4k(k^2+1)^2 - k \\ 2k(k^2+1) \end{pmatrix} = \begin{pmatrix} C_{3,k} \\ C_{2,k} \end{pmatrix}$$

More generally,

$$\mathcal{M}\begin{pmatrix} C_{n,k} \\ C_{n-1,k} \end{pmatrix} = \begin{pmatrix} C_{n+1,k} \\ C_{n,k} \end{pmatrix}$$

Theorem: 2.1

If $\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$ is a Pan-San sequence matrix, then the n^th^ term of the Pan-San sequence is generalized by

$C_{n,k} = \frac{1}{2\sqrt{k^2+2}}\left[\left((k^2+1) + k\sqrt{k^2+2}\right)^n - \left((k^2+1) - k\sqrt{k^2+2}\right)^n\right]$, where $n \in W$, the set of all whole numbers.

Proof:

Given

$$\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$$

The characteristic equation $\lambda^2 - 2(k^2+1)\lambda + 1 = 0$ of $\mathcal{M}$ reveals two distinct eigen values $\sigma = (k^2+1) + k\sqrt{k^2+2}$ and $\tau = (k^2+1) - k\sqrt{k^2+2}$.

Also, $\lambda^2 = 2(k^2+1)\lambda - 1$

$$\lambda^3 = \lambda^2.\lambda$$

$$= (4(k^2+1)^2 - 1)\lambda - 2(k^2+1) = \frac{1}{k}\left(C_{3,k}\lambda - C_{2,k}\right)$$

$$k\lambda^3 = C_{3,k}\lambda - C_{2,k}$$

$$\lambda^4 = [8(k^2+1)^3 - 4(k^2+1)]\lambda - [4(k^2+1) - 1] = \frac{1}{k}\left(C_{4,k}\lambda - C_{3,k}\right)$$

$$k\lambda^4 = C_{4,k}\lambda - C_{3,k}$$

In general, $k\lambda^n = C_{n,k}\lambda - C_{n-1,k}$ (1)

Since both σ and τ are the characteristic values, they must satisfy (1), hence

$k\sigma^n = \left(C_{n,k}\sigma - C_{n-1,k}\right)$ and $k\tau^n = \left(C_{n,k}\tau - C_{n-1,k}\right)$

$\Rightarrow k(\sigma^n - \tau^n) = C_{n,k}(\sigma - \tau)$

Therefore, $C_{n,k} = \frac{k(\sigma^n - \tau^n)}{(\sigma - \tau)}$

$C_{n,k} = \frac{1}{2\sqrt{k^2+2}}\left[\left((k^2+1) + k\sqrt{k^2+2}\right)^n - \left((k^2+1) - k\sqrt{k^2+2}\right)^n\right]$, where $n \in W$, the set of all whole numbers.

Theorem: 2.2

If $\{D_{n,k}\}$ and $\{C_{n,k}\}$ are Pan-San Buddy sequence and Pan-San sequence respectively, then

i. $kD_{n,k} = (k^2+1)C_{n,k} - C_{n-1,k}$

ii. $D_{n,k}C_{n,k} = \frac{1}{2}C_{2n,k}$

iii. $D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}$

iv. $\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = D_{2n-1,k}$

v. $C_{n+1,k} - C_{n-1,k} = 2kD_{n,k}$

vi. $D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}$

Proof:

i. By using the characteristic values of the Pan-San sequence as explained in theorem 2.1, their product is given by $\sigma\tau = 1$.

The closed form the Pan-San Buddy sequence is specified by

$$D_{n,k} = A\sigma^n + B\tau^n \tag{2}$$

The fundamental values $D_{0,k} = 1, D_{1,k} = k^2 + 1$ provides the subsequence system of linear equations $A + B = 1$

$$A\sigma + B\tau = k^2 + 1.$$

Precisely, $A = \frac{(k^2+1)-\tau}{\sigma-\tau}$ and $B = \frac{\sigma-(k^2+1)}{\sigma-\tau}$

Consequently, the specific value of $D_{n,k}$ is pointed out by

$$D_{n,k} = \frac{(k^2+1)-\tau}{\sigma-\tau}\sigma^n + \frac{\sigma-(k^2+1)}{\sigma-\tau}\tau^n$$

$$= \frac{1}{\sigma-\tau}[(k^2+1)(\sigma^n - \tau^n) + \sigma\tau(\tau^{n-1} - \sigma^{n-1})]$$

$$= \frac{(k^2+1)}{k}C_{n,k} - \frac{1}{k}C_{n-1,k}$$

$$\therefore \boldsymbol{kD_{n,k} = (k^2+1)C_{n,k} - C_{n-1,k}}$$

ii. The alternative forms of the above values of A and B are epitomized by

$$A = \frac{(k^2+1)-\tau}{\sigma-\tau} = \frac{k\sqrt{k^2+2}}{2k\sqrt{k^2+2}} = \frac{1}{2}$$

$$B = 1 - A = \frac{1}{2}$$

The equivalent values of the general term of the Pan-San Buddy sequence is noted as

$$\boldsymbol{D_{n,k} = \frac{1}{2}(\sigma^n + \tau^n)}$$

Hence,

$$D_{n,k}C_{n,k} = \left(\frac{\sigma^n+\tau^n}{2}\right)\left(\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)}\right)$$

$$= \frac{k}{2(\sigma-\tau)}(\sigma^{2n} - \tau^{2n})$$

$$\boldsymbol{D_{n,k}C_{n,k} = \frac{1}{2}C_{2n,k}}$$

iii. $$\frac{D_{n+1,k}-D_{n-1,k}}{C_{n,k}} = \frac{\frac{1}{2}(\sigma^{n+1}+\tau^{n+1})-\frac{1}{2}(\boldsymbol{\sigma^{n-1}+\tau^{n-1}})}{\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)}}$$

$$= \frac{(\sigma-\tau)}{2k}\frac{(\sigma^{n+1}+\tau^{n+1}-\sigma^{n-1}-\tau^{n-1})}{(\sigma^n-\tau^n)}$$

$$= \frac{(\sigma-\tau)}{2k}\frac{(\sigma^{n+1}+\tau^{n+1}-\sigma^n\tau-\sigma\tau^n)}{(\sigma^n-\tau^n)}$$

$$= \frac{(\sigma-\tau)}{2k}(\sigma - \tau)$$

$$= 2k(k^2 + 2)$$

$$\boldsymbol{D_{n+1,k} - D_{n-1,k} = 2k(k^2 + 2)C_{n,k}}$$

iv. $$\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = \left[\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)} + \frac{k(\sigma^{n-1}-\tau^{n-1})}{(\sigma-\tau)}\right]^2 + 1$$

$$= \left\{ \frac{k}{(\sigma - \tau)} [\sigma^n - \tau^n + \sigma^n \tau - \sigma \tau^n] \right\}^2 + 1$$

$$= \left\{ \frac{k}{(\sigma - \tau)} [\sigma^n (1 + \tau) - \tau^n (1 + \sigma)] \right\}^2 + 1$$

$$= \frac{k^2}{(\sigma - \tau)^2} [\sigma^{2n} (1 + \tau)^2 + \tau^{2n} (1 + \sigma)^2 - 2\sigma^n \tau^n (1 + \tau)(1 + \sigma)] + 1$$

$$= \frac{k^2}{(\sigma - \tau)^2} [\sigma^{2n} + \tau^{2n} + \sigma^2 \tau^2 (\sigma^{2n-2} + \tau^{2n-2}) + 2\sigma\tau(\sigma^{2n-1} + \tau^{2n-1}) - 4(k^2 + 2)] + 1$$

$$= \frac{k^2}{(\sigma - \tau)^2} [(\sigma^{2n} + \tau^{2n}) + \sigma^2 \tau^2 (\sigma^{2n-2} + \tau^{2n-2}) + 2\sigma\tau(\sigma^{2n-1} + \tau^{2n-1}) - 4(k^2 + 2)] + 1$$

$$= \frac{1}{4(k^2 + 2)} \left[2D_{2n,k} + 2D_{2n-2,k} + 4D_{2n-1,k} \right]$$

$$= \frac{1}{4(k^2 + 2)} 4(k^2 + 2\,) D_{2n-1,k}$$

$$\boldsymbol{\left(C_{n,k} + C_{n-1,k} \right)^2 + 1 = D_{2n-1,k}}$$

v. $C_{n+1,k} - C_{n-1,k} = 2(k^2 + 1\,) C_{n,k} - C_{n-1,k} - C_{n-1,k}$

$$= 2(k^2 + 1\,) C_{n,k} - 2C_{n-1,k}$$

$$\boldsymbol{C_{n+1,k} - C_{n-1,k} = 2kD_{n,k}}$$

vi. $D_{n+1,k} - D_{n-1,k} = \frac{1}{2}(\sigma^{n+1} + \tau^{n+1}) - \frac{1}{2}(\sigma^{n-1} + \tau^{n-1})$

$$= \frac{1}{2}[\sigma^{n+1} + \tau^{n+1} - \sigma^n \tau - \sigma \tau^n]$$

$$= \frac{1}{2}[\sigma^n (\sigma - \tau) - \tau^n (\sigma - \tau)]$$

$$= \frac{1}{2}[(\sigma^n - \tau^n)(\sigma - \tau)]$$

$$= \frac{\left(2k\sqrt{k^2+2}\right)^2}{2k} C_{n,k}$$

$$\boldsymbol{D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}}$$

3. PAN-SAN COMRADE AND PAN-SAN MATE SEQUENCES

The values of R and S in the world-wide equation $S^2 - dR^2 = 1$ for a firm non-zero square-free integer $d = k^2 - 2,\ k \in N - \{1\}$ creates two handsome sequences $0, k, 2k(k^2-1), 4k(k^2-1)^2 - k, 8k(k^2-1)^3 - 4k(k^2-1), \ldots$ and $1, k^2-1, 2(k^2-1)^2 - 1, 4(k^2-1)^3 - 3\,(k^2-1), \ldots$ And Called As Pan-San Comrade And Pan-San Mate Sequences. The nth term of the earlier sequence is construed by the relation

$R_{n,k} = 2(k^2-1)R_{n-1,k} - R_{n-2,k}$, where $R_{0,k} = 0, R_{1,k} = k,\quad k \in N - \{1\}$ and N is the set of all-natural numbers.

The nth term of the later sequence is inferred by the recurrence relation

$S_{n,k} = 2(k^2-1)S_{n-1,k} - S_{n-2,k}$, where $S_{0,k} = 1, S_{1,k} = k^2 - 1,\ k \in N - \{1\}$ and $n \in W$, the set of whole numbers.

Define the Pan-San Comrade sequence matrix as

$$\mathfrak{M} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathfrak{M}\begin{pmatrix} R_{1,k} \\ R_{0,k} \end{pmatrix} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k^2-1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2(k^2-1)^2 - 1 \\ k^2-1 \end{pmatrix} = \begin{pmatrix} R_{2,k} \\ R_{1,k} \end{pmatrix}$$

Also,

$$\mathfrak{M}\begin{pmatrix} R_{2,k} \\ R_{1,k} \end{pmatrix} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2(k^2-1)^2-1 \\ k^2-1 \end{pmatrix} = \begin{pmatrix} 4(k^2-1)^3-3(k^2-1) \\ 2(k^2-1)^2-1 \end{pmatrix} = \begin{pmatrix} R_{3,k} \\ R_{2,k} \end{pmatrix}$$

In general, $\mathfrak{M}\begin{pmatrix} R_{n,k} \\ R_{n-1,k} \end{pmatrix} = \begin{pmatrix} R_{n+1,k} \\ R_{n,k} \end{pmatrix}$

As to section, it is possible to prove the following theorem.

Theorem: 3.1

If $\mathfrak{M} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}$ is a Pan-San Comrade sequence matrix, then the n^th term of the Pan-San Comrade Sequence is hypothesized by

$R_{n,k} = \frac{1}{2\sqrt{k^2-2}}\left[\left((k^2-1)+k\sqrt{k^2-2}\right)^n - \left((k^2-1)-k\sqrt{k^2-2}\right)^n\right]$, where $n = 0,1,2,3, \ldots.$

Theorem:3.2

If $\{R_{n,k}\}$ and $\{S_{n,k}\}$ are Pan-San Comrade and Pan-San Mate sequences respectively, then

i. $kS_{n,k} = (k^2-1)R_{n,k} - R_{n-1,k}$

ii. $S_{n,k}R_{n,k} = \frac{1}{2}R_{2n,k}$

iii. $S_{n+1,k} - S_{n-1,k} = 2k(k^2-2)R_{n,k}$

iv. $(R_{n,k} - R_{n-1,k})^2 - 1 = S_{2n-1,k}$

v. $R_{n+1,k} - R_{n-1,k} = 2kS_{n,k}$

vi. $S_{n+1,k} - S_{n-1,k} = 2k(k^2-2)R_{n,k}$

4. CONCLUSION

In this paper, four disparate sequences and their recurrence relations named as Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are established by utilizing the generalized solutions (x,y) to the universal equation called as Pell equation for two non-zero square-free integers $d = k^2 + 2, d = k^2 - 2$ where $k \in N - \{1\}$. Also, the general formulae and few theorems are proved involving such sequences for distinct values of d and can analyze the corresponding results.

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