APPRAISAL OF INTEGER QUADRUPLES WITH GRACEFUL PROPERTY

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ABSTRACT:

In this article, an assortment of integer quadruples (p, q, r, s) where $p, q, r, s \in Z - \{0\}$, the set of all non-zero integers such that the arithmetic mean of any three members is a cube of an integer is scrutinized by applying various techniques.

KEYWORDS: Diophantine quadruples, Ternary quadratic Diophantine equation

INTRODUCTION:

In Number theory, a Diophantine equation is learning of solutions of polynomial equations or systems of equations in integers, rational numbers, or sometimes more general number rings [1-3]. Diophantus raised the problem of finding a set of four (rational) numbers which have the property that the product of any two numbers in the set increased by one is a square and found such a set$\left[\left(\frac{1}{16}\right), \left(\frac{33}{16}\right), \left(\frac{68}{16}\right), \left(\frac{105}{16}\right)\right]$of four positive rational numbers. Fermat first found a set of four positive integers with the above property, which was$(1,3,8,120)$. In this framework, one may go through [4-8]. In this communication, the collection of non-zero discrete Diophantine quadruples(p, q, r, s)in which the arithmetic mean of any three elements is a cubical integer is appraised by using different approaches.

PROCESS OF SCRUTINY:

Let p, q, r, s be four non-zero unequal integers such that the arithmetic mean of any three members is a cube of an integer. Commence the mathematical statement of this hypothesis as follows.

$$p + q + r = 3\alpha^3 \quad (1)$$

$$p + q + s = 3\beta^3 \quad (2)$$

$$p + r + s = 3\gamma^3 \quad (3)$$

$$q + r + s = 3\delta^3 \quad (4)$$

composed with the additional condition that

$$p + q + r + s = (\alpha + \beta + \gamma + \delta)z^3 \quad (5)$$

Also, by resolving the system of equations (1), (2), (3) and (4), the values of p, q, r, s in terms of four unknowns are stated by

$$p = \alpha^3 + \beta^3 + \gamma^3 - 2\delta^3 \quad (6)$$

$$q = \alpha^3 + \beta^3 + \delta^3 - 2\gamma^3 \quad (7)$$

$$r = \alpha^3 + \delta^3 + \gamma^3 - 2\beta^3 \quad (8)$$

$$s = \beta^3 + \gamma^3 + \delta^3 - 2\alpha^3 \quad (9)$$

Summing up the above four equations, the relation concerning the essential parameters is computed by

$$p + q + r + s = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 \quad (10)$$

Associating (5) and (10), the subsequent expression predictable by

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = (\alpha + \beta + \gamma + \delta)z^3 \quad (11)$$

Presenting the appropriate alterations

$$\alpha = m + n, \beta = 2m + n, \gamma = 2m - n \text{ and } \delta = m - n$$

where $m, n \in Z - \{0\}$ in (6), (7), (8) and (9), the equivalent choices of p, q, r, s in the necessary quadruples are reckoned by

$$p = 15m^3 + 9m^2n + 9mn^2 + 3n^3 \quad (12)$$

$$q = -6m^3 + 36m^2n + 3n^3 \quad (13)$$

$$r = -6m^3 - 36m^2n - 3n^3 \quad (14)$$

$$s = 15m^3 - 9m^2n + 9mn^2 - 3n^3 \quad (15)$$

Using the identical translations in (11) affords the following third-degree equation

$$z^3 = 3m^2 + 3n^2 \quad (16)$$

The procedure for veridiction of quadruples (p, q, r, s) where arithmetic mean of any three elements is a cubical integer by solving (16) in four different techniques is described below.

Technique -1:

Deliberating the choice of z as $z = 3a^2 + 3b^2$ where a, b are non- zero integers in (16) reduces the following expression

$$(\sqrt{3}m + i\sqrt{3}n)(\sqrt{3}m - i\sqrt{3}n) = \left[(\sqrt{3}a + i\sqrt{3}b)(\sqrt{3}a - i\sqrt{3}b)\right]^3$$

Equalizing the positive parts on both sides of the above equation stretches that

$$\left(\sqrt{3}m + i\sqrt{3}n\right) = \left(\sqrt{3}a + i\sqrt{3}b\right)^3 \quad (17)$$

Comparison of real and imaginary parts on two sides of (17) leads to

$$m = 3a^3 - 9ab^2$$

$$n = 9a^2b - 3b^3$$

Exchanging these values of m and n from (12) to (15), the needed quadruples (p, q, r, s) in two variables is inspected by

$$p = 15(3a^3 - 9ab^2)^3 + 9(3a^3 - 9ab^2)^2(9a^2b - 3b^3) + 9(3a^3 - 9ab^2)(9a^2b - 3b^3)^2 + 3(9a^2b - 3b^3)^3$$

$$q = -6(3a^3 - 9ab^2)^3 + 36(3a^3 - 9ab^2)^2(9a^2b - 3b^3) + 3(9a^2b - 3b^3)^3$$

$$r = -6(3a^3 - 9ab^2)^3 - 36(3a^3 - 9ab^2)^2(9a^2b - 3b^3) - 3(9a^2b - 3b^3)^3$$

$$s = 15(3a^3 - 9ab^2)^3 - 9(3a^3 - 9ab^2)^2(9a^2b - 3b^3) + 9(3a^3 - 9ab^2)(9a^2b - 3b^3)^2 - 3(9a^2b - 3b^3)^3$$

Numerical calculations for certain values of a and b fulfilling the hypothesis are tabularized below.

a	b	p	q	r	s	$\frac{p+q+r}{3}$	$\frac{p+q+s}{3}$	$\frac{p+r+s}{3}$	$\frac{q+r+s}{3}$
1	2	−609201	−20250	451494	−490293	$(-39)^3$	$(-72)^3$	$(-60)^3$	$(-27)^3$
2	1	180549	149283	−151875	−56457	$(39)^3$	$(45)^3$	$(-21)^3$	$(-27)^3$
3	1	8789472	8667000	−10556568	1848096	$(132)^3$	$(186)^3$	$(30)^3$	$(-24)^3$
3	2	3866697	11623986	−11387790	−13712571	$(111)^3$	$(84)^3$	$(-192)^3$	$(-165)^3$

Technique: 2

Consider the similar form of (16) as

$$3m^2 + 3n^2 = z^3 \times 1^2 \quad (18)$$

Implementing the conversions $z = \left(\sqrt{3}a\right)^2 + \left(\sqrt{3}b\right)^2$ and $1 = \frac{(1+i)(1-i)}{2}$ in (18) and factorizing on both sides, the relevant form of the above said equation progresses into

$$\left(\sqrt{3}m + i\sqrt{3}n\right)\left(\sqrt{3}m - i\sqrt{3}n\right) = \left(\frac{(1+i)(1-i)}{2}\right)^2 \left[\left(\sqrt{3}a + i\sqrt{3}b\right)\left(\sqrt{3}a - i\sqrt{3}b\right)\right]^3 \quad (19)$$

Expanding the positive parts on each side of (19) and correlating the real and imaginary parts, the corresponding values of m and n are pointed out by

$$m = -9a^2b + 3b^3$$

$$n = 3a^3 - 9ab^2$$

Thus, the options for p, q, r and s in the indispensable quadruples are calculated by

$$p = 15(-9a^2b + 3b^3)^3 + 9(-9a^2b + 3b^3)^2(3a^3 - 9ab^2)$$
$$+9(-9a^2b + 3b^3)(3a^3 - 9ab^2)^2 + 3(3a^3 - 9ab^2)^3$$

$$q = -6(-9a^2b + 3b^3)^3 + 36(-9a^2b + 3b^3)^2(3a^3 - 9ab^2) + 3(3a^3 - 9ab^2)^3$$

$$r = -6(-9a^2b + 3b^3)^3 - 36(-9a^2b + 3b^3)^2(3a^3 - 9ab^2) - 3(3a^3 - 9ab^2)^3$$

$$s = 15(-9a^2b + 3b^3)^3 - 9(-9a^2b + 3b^3)^2(3a^3 - 9ab^2)$$
$$+9(-9a^2b + 3b^3)(3a^3 - 9ab^2)^2 - 3(3a^3 - 9ab^2)^3$$

Numerical illustrations for positive values of a and b gratifying the supposition are offered in the following table below.

a	b	p	q	r	s	$\frac{p+q+r}{3}$	$\frac{p+q+s}{3}$	$\frac{p+r+s}{3}$	$\frac{q+r+s}{3}$
1	2	-56457	-151875	149283	180549	$(-27)^3$	$(-21)^3$	$(45)^3$	$(39)^3$
2	1	-490293	451494	-20250	-609201	$(-27)^3$	$(-60)^3$	$(-72)^3$	$(-39)^3$
2	3	-13712571	-11387790	11623986	3866697	$(-165)^3$	$(-192)^3$	$(84)^3$	$(111)^3$
3	2	-45013239	-2801385	34338249	-35639757	$(-165)^3$	$(-303)^3$	$(-249)^3$	$(-111)^3$

Technique: 3

Consider the equivalent form of (16) as given below

$$3m^2 + 3n^2 = z^3 \times 1 \quad (20)$$

By means of the replacements $z = \left(\sqrt{3}a\right)^2 + \left(\sqrt{3}b\right)^2$ and $1 = \frac{(4+3i)(4-3i)}{25}$ in (20), it gives

$$\left(\sqrt{3}m + i\sqrt{3}n\right)\left(\sqrt{3}m - i\sqrt{3}n\right) = \frac{(4+3i)(4-3i)}{25}\left[\left(\sqrt{3}a + i\sqrt{3}b\right)\left(\sqrt{3}a - i\sqrt{3}b\right)\right]^3$$

Ensuing the identical procedure as enlightened in previous techniques, the essential values of m and n are enumerated as

$$m = \left(\frac{12a^3-27a^2b-36ab^2+9b^3}{5}\right)$$

$$n = \left(\frac{9a^3+36a^2b-27ab^2-12b^3}{5}\right)$$

Since our objective and concentration is to treasure the integer values for the parameters, let us choose $a = 5u$ and $b = 5v$.

Consequently,

$$m = 300u^3 - 675u^2v - 900uv^2 + 225v^3$$

$$n = 225u^3 + 900u^2v - 675uv^2 - 300v^3$$

Thus, by implementing the above choices of two parametric values of m and n , the possibilities for p, q, r and s in the requisite quadruples are premeditated by

$$p = 15(300u^3 - 675u^2v - 900uv^2 + 225v^3)^3$$
$$+9(300u^3 - 675u^2v - 900uv^2 + 225v^3)^2(225u^3 + 900u^2v - 675uv^2 - 300v^3)$$
$$+9(300u^3 - 675u^2v - 900uv^2 + 225v^3)(225u^3 + 900u^2v - 675uv^2 - 300v^3)^2$$
$$+3(225u^3 + 900u^2v - 675uv^2 - 300v^3)^3$$

$$q = -6(300u^3 - 675u^2v - 900uv^2 + 225v^3)^3$$
$$+36(300u^3 - 675u^2v - 900uv^2 + 225v^3)^2(225u^3 + 900u^2v - 675uv^2 - 300v^3)$$
$$+3(225u^3 + 900u^2v - 675uv^2 - 300v^3)^3$$

$$r = -6(300u^3 - 675u^2v - 900uv^2 + 225v^3)^3$$
$$-36(300u^3 - 675u^2v - 900uv^2 + 225v^3)^2(225u^3 + 900u^2v - 675uv^2 - 300v^3)$$
$$-3(225u^3 + 900u^2v - 675uv^2 - 300v^3)^3$$

$$s = 15(300u^3 - 675u^2v - 900uv^2 + 225v^3)^3$$
$$-9(300u^3 - 675u^2v - 900uv^2 + 225v^3)^2(225u^3 + 900u^2v - 675uv^2 - 300v^3)$$
$$+9(300u^3 - 675u^2v - 900uv^2 + 225v^3)(225u^3 + 900u^2v - 675uv^2 - 300v^3)^2$$
$$-3(225u^3 + 900u^2v - 675uv^2 - 300v^3)^3$$

Mathematical calculations for the elements in the quadruples satisfying the necessary conditions few values of a and b are given in the following table:

a	b	Values of p,q,r,s	$\frac{p+q+r}{3}$	$\frac{p+q+s}{3}$	$\frac{p+r+s}{3}$	$\frac{q+r+s}{3}$
1	1	p = -16078500000 q = 12909375000 r = 982125000 s = -19075500000	$(-900)^3$	$(-1950)^3$	$(-2250)^3$	$(-1200)^3$
1	2	p = -901791984375 q = -847494140625 r = 1125283640625 s = -277755328125	$(-5925)^3$	$(-8775)^3$	$(-2625)^3$	$(225)^3$
2	2	p = -8232192000000 q = 6609600000000 r = 502848000000 s = -9766656000000	$(-7200)^3$	$(-15600)^3$	$(-18000)^3$	$(-9600)^3$
3	2	p = -26675589515625 q = 90423748828125 r = -63754377328125 s = -72566871046875	$(-1275)^3$	$(-14325)^3$	$(-37875)^3$	$(-24825)^3$

Technique: 4

Modify the cubic equation (16) as

$$(\sqrt{3}m)^2 + (\sqrt{3}n)^2 = z^3 \quad (21)$$

This implies that

$$X^2 + Y^2 = z^3 \quad (22)$$

where $X = \sqrt{3}m$ and $Y = \sqrt{3}n$ (23)

Retain the successive values of X, Y and z in (22) as specified below

$$X = u^3 + uv^2$$

$$Y = v^3 + u^2v$$

$$z = u^2 + v^2$$

The equivalent values of m and n are articulated from (23) as follows.

$$m = \left(\frac{u^3+uv^2}{\sqrt{3}}\right) \quad (24)$$

$$n = \left(\frac{v^3+vu^2}{\sqrt{3}}\right) \quad (25)$$

Case(i):

To exhibit the options for m and n in integer, swap the alterations $u = 3\sqrt{3}a$ and $v = \sqrt{3}b$ in (24) and (25), it is perceived that

$$m = 81a^3 + 9ab^2$$

$$n = 3b^3 + 27a^2b$$

Subsequently,

$$p = 15(81a^3 + 9ab^2)^3 + 9(81a^3 + 9ab^2)^2(3b^3 + 27a^2b) + 9(81a^3 + 9ab^2)(3b^3 + 27a^2b)^2 + 3(3b^3 + 27a^2b)^3$$

$$q = -6(81a^3 + 9ab^2)^3 + 36(81a^3 + 9ab^2)^2(3b^3 + 27a^2b) + 3(3b^3 + 27a^2b)^3$$

$$r = -6(81a^3 + 9ab^2)^3 - 36(81a^3 + 9ab^2)^2(3b^3 + 27a^2b) - 3(3b^3 + 27a^2b)^3$$

$$s = 15(81a^3 + 9ab^2)^3 - 9(81a^3 + 9ab^2)^2(3b^3 + 27a^2b) + 9(81a^3 + 9ab^2)(3b^3 + 27a^2b)^2 - 3(3b^3 + 27a^2b)^3$$

Here, some arithmetical samples are neatly stated in the subsequent table:

a	b	p	q	r	s	$\frac{p+q+r}{3}$	$\frac{p+q+s}{3}$	$\frac{p+r+s}{3}$	$\frac{q+r+s}{3}$
1	1	13932000	4455000	−13203000	9396000	$(120)^3$	$(210)^3$	$(150)^3$	$(60)^3$
1	2	41463981	30252690	−49472046	19397313	$(195)^3$	$(312)^3$	$(156)^3$	$(39)^3$
2	1	4952191851	4102893	−3549002445	4057761177	$(777)^3$	$(1443)^3$	$(1221)^3$	$(555)^3$
2	2	7133184000	228096000	−6759936000	4810752000	$(960)^3$	$(1680)^3$	$(1200)^3$	$(480)^3$

Case (ii):

To enumerate an integer solution, exchange the values of u and v in single parameter as $u = \sqrt{3}k^2$ and $v = \sqrt{3}k$ in (24) and (25). it is exposed that

$$m = 3k^6 + 3k^4$$

$$n = 3k^3 + 3k^5$$

Using the above replacements of m and n in (12), (13), (14) and 15), the values of the required p, q, r, s in only one parameter are premeditated by

$$p = 15(3k^6 + 3k^4)^3 + 9(3k^6 + 3k^4)^2(3k^3 + 3k^5) + 9(3k^6 + 3k^4)(3k^3 + 3k^5)^2$$
$$+3(3k^3 + 3k^5)^3$$
$$q = -6(3k^6 + 3k^4)^3 + 36(3k^6 + 3k^4)^2(3k^3 + 3k^5) + 3(3k^3 + 3k^5)^3$$
$$r = -6(3k^6 + 3k^4)^3 - 36(3k^6 + 3k^4)^2(3k^3 + 3k^5) - 3(3k^3 + 3k^5)^3$$
$$s = 15(3k^6 + 3k^4)^3 - 9(3k^6 + 3k^4)^2(3k^3 + 3k^5) + 9(3k^6 + 3k^4)(3k^3 + 3k^5)^2 -$$
$$3(3k^3 + 3k^5)^3$$

Numerical specimens for limited values of k are given in the ensuing table:

k	Values of $p, q, r\ s$	$\frac{p+q+r}{3}$	$\frac{p+q+s}{3}$	$\frac{p+r+s}{3}$	$\frac{q+r+s}{3}$
2	p = 305856000 q = 171072000 r = -336960000 s = 171072000	$(360)^3$	$(600)^3$	$(360)^3$	$(120)^3$
3	p = 274223556000 q = 87687765000 r = -259874649000 s = 184941468000	$(3240)^3$	$(5670)^3$	$(4050)^3$	$(1620)^3$
4	p = 39746297659392 q = 6780864430080 r = -33487038185472 s = 29522840518656	$(16320)^3$	$(29376)^3$	$(22848)^3$	$(9792)^3$
5	p = 1990893937500000 q = 141809484375000 r = -1532098546875000 s = 1568246062500000	$(58500)^3$	$(107250)^3$	$(87750)^3$	$(39000)^3$

Python Program for numerical verifications for all other choices of the parameters substantial our propositions are exemplified below:

```
import math
while True:
    c = input("Enter choice(1/2/3/4/5): ")
    if c in ('1','2','4'):
        a = int(input('Enter a Number a : '))
        b = int(input('Enter the second number b : '))
```

```
if c in ('3'):
    a = int(input('Enter a Number a : '))
    v = int(input('Enter the second number v : '))
if c in ('5'):
    k = int(input('Enter a Number k : '))
if (c == '1'):
    m = 3 * a ** 3 − 9 * a * b ** 2
    n = 9 * a ** 2 * b − 3 * b ** 3
elif(c == '2'):
    m = −9 * a ** 2 * b + 3 * b ** 3
    n = 3 * a ** 3 − 9 * a * b ** 2
elif(c == '3'):
    m = 300 * u ** 3 − 675 * u ** 2 * v − 900 * u * v ** 2 + 225 * v ** 3
    n = 225 * u ** 3 + 900 * u ** 2 * v − 675 * u * v ** 2 − 300 * v ** 3
elif(c == '4'):
    m = 81 * a ** 3 + 9 * a * b ** 2
    n = 3 * b ** 3 + 27 * a ** 2 * b
elif(c == '5'):
    m = 3 * k ** 6 + 3 * k ** 4
    n = 3 * k ** 3 + 3 * k ** 5
else:
    print('Invalid Input')
    break
p = 15 * m ** 3 + 9 * m ** 2 * n + 9 * m * n ** 2 + 3 * n ** 3
q = −6 * m ** 3 + 36 * m ** 2 * n + 3 * n ** 3
r = −6 * m ** 3 − 36 * m ** 2 * n − 3 * n ** 3
s = 15 * m ** 3 − 9 * m ** 2 * n + 9 * m * n ** 2 − 3 * n ** 3
a1 = (p + q + r)/3
a2 = (p + q + s)/3
a3 = (p + r + s)/3
a4 = (q + r + s)/3
if (a1 < 0):
    a11 = −1 * a1
    x = pow(a11,1/3)
else:
    x = pow(a1,1/3)
if (x == 0):
    x = 0
else:
    x+= 1
if (a1 < 0):
    x = −x
if (a2 < 0):
```

```
        a21 = -1 * a2
        y = pow(a21,1/3)
    else:
        y = pow(a2,1/3)
    if (y == 0):
        y = 0
    else:
        y+= 1
    if (a2 < 0):
        y = -y
    if (a3 < 0):
        a31 = -1 * a3
        z = pow(a31,1/3)
    else:
        z = pow(a3,1/3)
    if (z == 0):
        z = 0
    else:
        z+= 1
    if (a3 < 0):
        z = -z
    if (a4 < 0):
        a41 = -1 * a4
        xx = pow(a41,1/3)
    else:
        xx = pow(a4,1/3)
    if (xx == 0):
        xx = 0
    else:
        xx+= 1
    if (a4 < 0):
        xx = -xx
    print('p : ',p)
    print('q : ',q)
    print('r : ',r)
    print('s : ',s)
    print('(p + q + r)/3 : ',int(x),'^3')
    print('(p + q + s)/3 : ',int(y),'^3')
    print('(p + r + s)/3 : ',int(z),'^3')
    print('(q + r + s)/3 : ',int(xx),'^3')
```

CONCLUSION:

In this communication, the collection of distinct quadruples (p, q, r, s) among them the members are non-zero integers in order that the arithmetic mean of any three members is a cube by applying dissimilar techniques is discovered. To conclude, one can pursuit for unlike quintuples, sextuples etc such that the elements satisfying an extraordinary condition.

Note:

It is possible to find infinite number of quadruples (p, q, r, s) satisfying the needed condition by trading 1 as follows

$$1 = \left(\frac{(8+6i)(8-6i)}{100}\right)$$

$$1 = \left(\frac{(12+5i)(12-5i)}{169}\right)$$

$$1 = \left(\frac{(12+9i)(12-9i)}{225}\right) \text{ etc}$$

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