THE ANTI-DERIVATIVE FUNCTIONS KNOWN AS BANGABANDHU FUNCTIONS

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Dedication: This paper is dedicated to the greatest Bengali of the millennium, the great architect of independent Bangladesh, the father of the Bengali nation, Bangabandhu Sheikh Mujibur Rahman, and the functions are named after him.

ABSTRACT: We can evaluate every definite integral using the limit or summation or the Trapezoidal rule and any other established rules. With the help of anti-derivative and the Fundamental Theorem of Calculus, we can also compute the value of various definite integrals. But every definite integral can't be evaluated by anti-derivative because every function has no anti-derivative formula as a function of the real variable. In that case, the function should be integrable within the interval. An attempt has been made here to establish general anti-derivative formulae named "Bangabandhu functions," for such types of particular functions which have no elementary functions for anti-derivative in this paper. From these formulae, we can easily calculate the area bounded by the graph of the given function and other boundaries up to certain decimal places.

Keywords **:** Integral, anti-derivative, generalized, Gaussian, approximated

1. INTRODUCTION

The purpose of integration is to determine the area of plane surfaces, area of curved surfaces, and the volume of solids, which sometimes can be calculated easily with the help of anti-derivatives and the Fundamental Theorem of Calculus concerning some conditions. Archimedes, the great scientist of antiquity, solved some problems in determining the area and volume of solids in the 3rd century BC, which are considered the basis of Integral Calculus. Later, in the hands of world-famous scientists like Isaac Newton, Gottfried Wilhelm Leibniz, Bernhard Riemann, Integral Calculus has become an essential modern tool of science today. If we can memorize a formula that is derivative of a function or can find an anti-derivative to that function by applying techniques, then we can easily find the respective area bounded by that function. And if the function intertwined with the field is not integrable, then the surprisingly Fundamental Theorem of Calculus, cannot be used to determine the area. However, it is possible to find the area by more than one established rule (Limit, Summation, the Trapezoidal, the Simpson, etc.) Our main objective of this paper is to establish general anti-derivative formulae for the functions like G*(x)* = e^{x^n}, e^{-x^n}, $n \in$ N and how to solve $\int_a^b e^{x^n}\, dx$, $\int_a^b e^{-x^n}\, dx$ through a simple method (using expansion of a function in a series) without using error functions erf(x), erfi(x), erfc(x), etc., and Gamma function. [2, 5]

2. RELATED WORKS

In the past, Toyesh Prakash Sharma [8], in most of the books [1] , [see page 712 [7]] used a similar process to evaluate the indefinite and definite integral. Here has been tried to express the work into a simple form using expansion of a function, summation of series, and mathematical induction rule along with the Fundamental Theorem of Calculus.

3. MATHEMATICAL BACKGROUND

3.1 Anti-derivative: A function F(x) is said to be anti-derivative of another function G(x) in an interval if $\mathrm{F}'(x) = \mathrm{G}(x)$ holds and it is generally denoted by $\int \mathrm{G}(x)dx = \mathrm{F}(x)$. Again, since $\frac{d}{dx}\{\mathrm{F}(x) + \mathrm{c}\} = \mathrm{F}'(x)$ for any real value of c, so the anti-derivative or indefinite integral of G(x) is $\int \mathrm{G}(x)dx = \mathrm{F}(x) + \mathrm{c}$, where c is an arbitrary constant which is called constant of integration.
Example 1:Let $\mathrm{G}(x) = 2x$ and $\mathrm{F}(x) = x^2$, now $\mathrm{F}'(x) = 2x = \mathrm{G}(x)$ so, $\int 2x\, dx = x^2 + \mathrm{c}$

3.2 Summation of series: If an infinite series is S_∞ = a + (a+d) + (a+2d) + ……………∞, then the m-th term of the series is denoted by {a + (m-1).d} and the summation of the series is $S_m = \sum_{m=1}^{\infty}\{a + (m-1).d\}$

3.3 Fundamental Theorem of Calculus: If $y = \mathrm{G}(x)$ is an integrable function in the closed interval [a, b] and the anti-derivative of $\mathrm{G}(x)$ in the open interval (a, b) is $\mathrm{F}(x) + \mathrm{c}$, then the area or definite integral $\int_a^b \mathrm{G}(x)\, dx = \mathrm{F}(b) - \mathrm{F}(a)$
Example 2: $\int_0^1 e^x\, dx = [e^x + \mathrm{c}]_0^1 = (e^1 + \mathrm{c}) - (e^0 + \mathrm{c}) = e - 1$

4. MATHEMATICAL INDUCTIONS

4.1 Induction1: Let $B_1(x) = \int e^x\, dx$
We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ \infty$

$$\therefore B_1(x) = \int e^x\, dx = \int(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ \infty)\, dx$$

$$= \left(x + \frac{x^2}{2.1!} + \frac{x^3}{3.2!} + \frac{x^4}{4.3!} + \frac{x^5}{5.4!} + \frac{x^6}{6.5!} + \frac{x^7}{7.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ \infty\right) + \mathrm{c}$$

$$= (\frac{x^1}{1.0!} + \frac{x^2}{2.1!} + \frac{x^3}{3.2!} + \frac{x^4}{4.3!} + \frac{x^5}{5.4!} + \frac{x^6}{6.5!} + \frac{x^7}{7.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ \infty) + \mathrm{c}$$

$$= \sum_{m=1}^{\infty}\frac{x^m}{m.(m-1)!} + \mathrm{c} \quad \textbf{(3.2)}$$

Therefore, the anti-derivative of e^x i.e., $B_1(x) = \int e^x\, dx = \sum_{m=1}^{\infty}\frac{x^m}{m.(m-1)!} + \mathrm{c}$
Similarly, we can prove that the anti-derivative of e^{-x}
i.e., $B_{1*}(x) = \int e^{-x}\, dx = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\, x^m}{m.(m-1)!} + \mathrm{c}$

Note: $B_1(x) = (\frac{x^1}{1.0!} + \frac{x^2}{2.1!} + \frac{x^3}{3.2!} + \frac{x^4}{4.3!} + \frac{x^5}{5.4!} + \frac{x^6}{6.5!} + \frac{x^7}{7.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ \infty) + \mathrm{c}$

$= (1 + \frac{x^1}{1.0!} + \frac{x^2}{2.1!} + \frac{x^3}{3.2!} + \frac{x^4}{4.3!} + \frac{x^5}{5.4!} + \frac{x^6}{6.5!} + \frac{x^7}{7.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ \infty) + (\mathrm{c} - 1)$

$= (1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ \infty) + (\mathrm{c} - 1)$

$= (1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty) + c$ [Since c is an arbitrary constant of integration]

$= e^x + c$

$\therefore \int e^x dx = e^x + c$, Therefore, there is no contradiction with the usual anti-derivative of e^x

4.2 Induction2: Let $B_2(x) = \int e^{x^2} dx$

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty$

$$\therefore B_2(x) = \int e^{x^2} dx = \int (1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty)\, dx$$

$$= \left(x + \frac{x^3}{3.1!} + \frac{x^5}{5.2!} + \frac{x^7}{7.3!} + \frac{x^9}{9.4!} + \frac{x^{11}}{11.5!} + \frac{x^{13}}{13.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty\right) + c$$

$$= (\frac{x^1}{1.0!} + \frac{x^3}{3.1!} + \frac{x^5}{5.2!} + \frac{x^7}{7.3!} + \frac{x^9}{9.4!} + \frac{x^{11}}{11.5!} + \frac{x^{13}}{13.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty) + c$$

$$= \sum_{m=1}^{\infty} \frac{x^{2m-1}}{(2m-1).(m-1)!} + c \quad \textbf{(3.2)}$$

Therefore, the anti-derivative of e^{x^2} i.e., $B_2(x) = \int e^{x^2}\, dx = \sum_{m=1}^{\infty} \frac{x^{2m-1}}{(2m-1).(m-1)!} + c$ which is free of erfi(x) [5]

Similarly, we can prove that the anti-derivative of e^{-x^2}

i.e., $B_{2*}(x) = \int e^{-x^2} dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{2m-1}}{(2m-1).(m-1)!} + c$ which is free of erf(x) [5]

4.3 Induction3: Let $B_3(x) = \int e^{x^3} dx$

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty$

$$\therefore B_3(x) = \int e^{x^3} dx = \int (1 + \frac{x^3}{1!} + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \frac{x^{15}}{5!} + \frac{x^{18}}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty)\, dx$$

$$= \left(x + \frac{x^4}{4.1!} + \frac{x^7}{7.2!} + \frac{x^{10}}{10.3!} + \frac{x^{13}}{13.4!} + \frac{x^{16}}{16.5!} + \frac{x^{19}}{19.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty\right) + c$$

$$= (\frac{x^1}{1.0!} + \frac{x^4}{4.1!} + \frac{x^7}{7.2!} + \frac{x^{10}}{10.3!} + \frac{x^{13}}{13.4!} + \frac{x^{16}}{16.5!} + \frac{x^{19}}{19.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty) + c$$

$$= \sum_{m=1}^{\infty} \frac{x^{3m-2}}{(3m-2).(m-1)!} + c \quad \textbf{(3.2)}$$

Therefore, the anti-derivative of e^{x^3} i.e., $B_3(x) = \int e^{x^3}\, dx = \sum_{m=1}^{\infty} \frac{x^{3m-2}}{(3m-2).(m-1)!} + c$

Similarly, we can prove that the anti-derivative of e^{-x^3}

i.e., $B_{3*}(x) = \int e^{-x^3}\, dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{3m-2}}{(3m-2).(m-1)!} + c$

4.4 Induction4: Let $B_4(x) = \int e^{x^4} dx$

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty$

$$\therefore B_4(x) = \int e^{x^4} dx = \int (1 + \frac{x^4}{1!} + \frac{x^8}{2!} + \frac{x^{12}}{3!} + \frac{x^{16}}{4!} + \frac{x^{20}}{5!} + \frac{x^{24}}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty)\, dx$$

$$= \left(x + \frac{x^5}{5.1!} + \frac{x^9}{9.2!} + \frac{x^{13}}{13.3!} + \frac{x^{17}}{17.4!} + \frac{x^{21}}{21.5!} + \frac{x^{25}}{25.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty\right) + c$$

$$= (\frac{x^1}{1.0!} + \frac{x^5}{5.1!} + \frac{x^9}{9.2!} + \frac{x^{13}}{13.3!} + \frac{x^{17}}{17.4!} + \frac{x^{21}}{21.5!} + \frac{x^{25}}{25.6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \infty) + c$$

$$= \sum_{m=1}^{\infty} \frac{x^{4m-3}}{(4m-3).(m-1)!} + c \quad \textbf{(3.2)}$$

Therefore, the anti-derivative of e^{x^4} i.e., $B_4(x) = \int e^{x^4}\,dx = \sum_{m=1}^{\infty} \frac{x^{4m-3}}{(4m-3).(m-1)!} + c$

Similarly, we can prove that the anti-derivative of e^{-x^4}

i.e., $B_{4*}(x) = \int e^{-x^4}\,dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{4m-3}}{(4m-3).(m-1)!} + c$

5. MAIN RESULT

From the above results, we can decide with the help of the mathematical induction rule that The anti-derivative Bangabandhu functions as follows:

$B_n(x) = \int e^{x^n} dx = \sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c$, where $n \in N$

And

$B_{n*}(x) = \int e^{-x^n} dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c$, where $n \in N$ [Generalized error functions [2]]

6. DEFINITE INTEGRAL $\int_0^1 e^{x^n}\,dx$ AND $\int_0^1 e^{-x^n}\,dx, n \in N$ BY BANGABANDHU FUNCTIONS

6.1 $\int_0^1 e^{x^n}\,dx$ for n = 1, 2, 3, 4

6.1.1 $\int_0^1 e^{x}\,dx$

We have Bangabandhu function

$B_n(x) = \int e^{x^n}\,dx = \sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c$, where $n \in N$

$\therefore [\,B_n(x)\ ]_0^1 = \int_0^1 e^{x^n}\,dx = \left[\sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c\ \right]_0^1 = \sum_{m=1}^{\infty} \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\,B_1(x)\ ]_0^1 = \int_0^1 e^{x}\,dx = \sum_{m=1}^{\infty} \frac{1}{\{(1.m-(1-1)\}.(m-1)!} = \sum_{m=1}^{\infty} \frac{1}{m.(m-1)!}$

$= \frac{1}{1.0!} + \frac{1}{2.1!} + \frac{1}{3.2!} + \frac{1}{4.3!} + \frac{1}{5.4!} + \frac{1}{6.5!} + \frac{1}{7.6!} + \cdots \dots\dots\dots\dots\dots\dots\ \infty$

= 1+ 0.5 + 0.167 + 0.042 + 0.008 + 0.001 + 0.000 + ………..

≈ 1.718 (Approximated up to three decimal places), this result matches with the Trapezoidal rule and other existing rules.

6.1.2 $\int_0^1 e^{x^2}\,dx$

$[\,B_n(x)\,]_0^1 = \int_0^1 e^{x^n}\,dx = \left[\sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c\ \right]_0^1 = \sum_{m=1}^{\infty} \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\,B_2(x)\ ]_0^1 = \int_0^1 e^{x^2}\,dx = \sum_{m=1}^{\infty} \frac{1}{\{(2.m-(2-1)\}.(m-1)!} = \sum_{m=1}^{\infty} \frac{1}{(2m-1).(m-1)!}$

$= \frac{1}{1.0!} + \frac{1}{3.1!} + \frac{1}{5.2!} + \frac{1}{7.3!} + \frac{1}{9.4!} + \frac{1}{11.5!} + \frac{1}{13.6!} + \cdots \dots\dots\dots\dots\dots\dots\ \infty$

= 1+ 0.333 + 0.1 + 0.024 + 0.005 + 0.001 + 0.000 + ………..

≈ 1.463 (Approximated up to three decimal places)

6.1.3 $\int_0^1 e^{x^3}\,dx$

$[\,B_n(x)\,]_0^1 = \int_0^1 e^{x^n}\,dx = \left[\sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + \text{c}\right]_0^1$

$= \sum_{m=1}^{\infty} \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\,B_3(x)\quad]_0^1 = \int_0^1 e^{x^3}\,dx = \sum_{m=1}^{\infty} \frac{1}{\{(3.m-(3-1)\}.(m-1)!} = \sum_{m=1}^{\infty} \frac{1}{(3m-2).(m-1)!}$

$= \frac{1}{1.0!} + \frac{1}{4.1!} + \frac{1}{7.2!} + \frac{1}{10.3!} + \frac{1}{13.4!} + \frac{1}{16.5!} + \frac{1}{19.6!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots$

= 1+ 0.25 + 0.071 + 0.017 + 0.003 + 0.001 + 0.000 + ………..

≈ 1.342 (Approximated up to three decimal places)

6.1.4 $\int_0^1 e^{x^4}\,dx$

$[\,B_n(x)\,]_0^1 = \int_0^1 e^{x^n}\,dx = \left[\sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + \text{c}\right]_0^1$

$= \sum_{m=1}^{\infty} \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\,B_4(x)\quad]_0^1 = \int_0^1 e^{x^4}\,dx = \sum_{m=1}^{\infty} \frac{1}{\{(4.m-(4-1)\}.(m-1)!} = \sum_{m=1}^{\infty} \frac{1}{(4m-3).(m-1)!}$

$= \frac{1}{1.0!} + \frac{1}{5.1!} + \frac{1}{9.2!} + \frac{1}{13.3!} + \frac{1}{17.4!} + \frac{1}{21.5!} + \frac{1}{25.6!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \infty$

= 1+ 0.2 + 0.056 + 0.013 + 0.003 + 0.001 + 0.000 + ………..

≈ 1.273 (Approximated up to three decimal places)

6.2 $\int_0^1 e^{-x^n}\,dx$ **for n= 1, 2, 3, 4**

6.2.1 $\int_0^1 e^{-x}\,dx$

We have Bangabandhu function

$B_{n*}(x) = \int e^{-x^n}\,dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ \ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!}$ + c, where $n \in$ N

$\therefore [\,B_{n*}(x)\quad]_0^1 = \int_0^1 e^{-x^n}\,dx = [\sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + \text{c}\ ]_0^1$

$= \sum_{m=1}^{\infty}(-1)^{m-1}\ \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\,B_{1*}(x)\quad]_0^1 = \int_0^1 e^{-x}\,dx = \sum_{m=1}^{\infty}(-1)^{m-1}\ \frac{1}{\{(1.m-(1-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1}\ \frac{1}{m.(m-1)!}$

$= \frac{1}{1.0!} - \frac{1}{2.1!} + \frac{1}{3.2!} - \frac{1}{4.3!} + \frac{1}{5.4!} - \frac{1}{6.5!} + \frac{1}{7.6!} - \cdots \ldots \ldots \ldots \ldots \ldots \ldots \infty$

= 1- 0.5 + 0.167 - 0.042 + 0.008 - 0.001 + 0.000 - ………..

≈ 0.632 (Approximated up to three decimal places), this result matches with the Trapezoidal rule and other existing rules.

6.2.2 $\int_0^1 e^{-x^2}\,dx$

$$[\,B_{n*}(x)\,]_0^1 = \int_0^1 e^{-x^n}\,dx = \left[\sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ \ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + \text{c}\right]_0^1$$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\ B_{2*}(x)\ ]_0^1 = \int_0^1 e^{-x^2}\,dx = \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{\{(2.m-(2-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{(2m-1).(m-1)!}$

$= \frac{1}{1.0!} - \frac{1}{3.1!} + \frac{1}{5.2!} - \frac{1}{7.3!} + \frac{1}{9.4!} - \frac{1}{11.5!} + \frac{1}{13.6!} - \cdots \ldots \ldots \ldots \ldots \ldots \ldots$

= 1- 0.333 + 0.1 - 0.024 + 0.005 - 0.001 + 0.000 - ………..

≈ 0.747 (Approximated up to three decimal places) which matches with [see page 715 [1]]

6.2.3 $\int_0^1 e^{-x^3}\,dx$

$$[\ B_{n*}(x)\ ]_0^1 = \int_0^1 e^{-x^n}\,dx = \left[\sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + \text{c}\right]_0^1$$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\ B_{3*}(x)\ ]_0^1 = \int_0^1 e^{-x^3}\,dx = \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{\{3.m-(3-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{(3m-2).(m-1)!}$

$= \frac{1}{1.0!} - \frac{1}{4.1!} + \frac{1}{7.2!} - \frac{1}{10.3!} + \frac{1}{13.4!} - \frac{1}{16.5!} + \frac{1}{19.6!} - \cdots \ldots \ldots \ldots \ldots \ldots \ldots$

= 1- 0.25 + 0.071 - 0.017 + 0.003 - 0.001 + 0.000 - ………..

≈ 0.806 (Approximated up to three decimal places)

6.2.4 $\int_0^1 e^{-x^4}\,dx$

$$[\ B_{n*}(x)\ ]_0^1 = \int_0^1 e^{-x^n}\,dx = \left[\sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + \text{c}\right]_0^1$$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{\{(n.m-(n-1)\}.(m-1)!}$

Therefore, $[\ B_{4*}(x)\ ]_0^1 = \int_0^1 e^{-x^4}\,dx = \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{\{4.m-(4-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \ \frac{1}{(4m-3).(m-1)!}$

$= \frac{1}{1.0!} - \frac{1}{5.1!} + \frac{1}{9.2!} - \frac{1}{13.3!} + \frac{1}{17.4!} - \frac{1}{21.5!} + \frac{1}{25.6!} - \cdots \ldots \ldots \ldots \ldots \ldots \ldots$

= 1- 0.2 + 0.056 - 0.013 + 0.003 - 0.001 + 0.000 - ………..

≈ 0.845 (Approximated up to three decimal places)

7. $\int_a^b e^{-x^2}\,dx$ AS PROPER INTEGRAL

7.1 Evaluation of $\int_1^2 e^{-x^2}\,dx$ by Bangabandhu function:

We know that the Bangabandhu function is

$B_{n*}(x) = \int e^{-x^n}dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + \text{c}$, where $n \in \text{N}$

Now by putting n = 2 in this function we get

$\int_1^2 e^{-x^2}\,dx = [\,B_{2*}(x)\,]_1^2 = B_{2*}(2) - B_{2*}(1)$

$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 2^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!} - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 1^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1}\ \frac{2^{2m-1}}{(2m-1).(m-1)!} - \sum_{m=1}^{\infty}(-1)^{m-1}\ \frac{1}{(2m-1).(m-1)!}$

$=(\frac{2}{1.0!} - \frac{8}{3.1!} + \frac{32}{5.2!} - \frac{128}{7.3!} + \frac{512}{9.4!} - \frac{2048}{11.5!} + \frac{8192}{13.6!} - \frac{32768}{15.7!} + \frac{131072}{17.8!} - \frac{524288}{19.9!} + \frac{2097152}{21.10!} - \frac{8388608}{23.11!} + \frac{33554432}{25.12!} - \frac{134217728}{27.13!} + \frac{536870912}{29.14!} - \dots\dots\dots\dots\dots\dots \infty) - (\frac{1}{1.0!} - \frac{1}{3.1!} + \frac{1}{5.2!} - \frac{1}{7.3!} + \frac{1}{9.4!} - \frac{1}{11.5!} + \frac{1}{13.6!} - \cdots \dots\dots\dots\dots\dots\dots \infty)$

= (2 – 2.667 + 3.2 – 3.048 + 2.370 – 1.552 + 0.875 – 0.433 + 0.191 – 0.076 + 0.028 – 0.009 + 0.003 – 0.001 + 0.000 - …….) – (1- 0.333 + 0.1 - 0.024 + 0.005 - 0.001 + 0.000 - ………..)

= 0.881 – 0.747 [**6.2.2]**

≈ 0.134 (Approximated up to three decimal places)

7.2 Evaluation of $\int_2^3 e^{-x^2}\,dx$ by Bangabandhu function:

We know that the Bangabandhu function is

$B_{n*}(x) = \int e^{-x^n} dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!}$ + c, where $n \in$ N

Now by putting n = 2 in this function we get

$\int_2^3 e^{-x^2}\,dx = [\,B_{2*}(x)\,]_2^3 = B_{2*}(3) - B_{2*}(2)$

$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 3^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!} - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 2^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1}\ \frac{3^{2m-1}}{(2m-1).(m-1)!} - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 2^{2m-1}}{(2m-1).(m-1)!}$

$=(\frac{3}{1.0!} - \frac{3^3}{3.1!} + \frac{3^5}{5.2!} - \frac{3^7}{7.3!} + \frac{3^9}{9.4!} - \frac{3^{11}}{11.5!} + \frac{3^{13}}{13.6!} - \frac{3^{15}}{15.7!} + \frac{3^{17}}{17.8!} - \frac{3^{19}}{19.9!} + \frac{3^{21}}{21.10!} - \frac{3^{23}}{23.11!} + \frac{3^{25}}{25.12!} - \frac{3^{27}}{27.13!} + \frac{3^{29}}{29.14!} - \frac{3^{31}}{31.15!} + \frac{3^{33}}{33.16!} - \frac{3^{35}}{35.17!} + \frac{3^{37}}{37.18!} - \frac{3^{39}}{39.19!} + \frac{3^{41}}{41.20!} - \frac{3^{43}}{43.21!} + \frac{3^{45}}{45.22!} - \frac{3^{47}}{47.23!} + \frac{3^{49}}{49.24!} - \frac{3^{51}}{51.25!} + \frac{3^{53}}{53.26!} - \frac{3^{55}}{55.27!} + \cdots \dots\dots\dots\dots\dots \infty - \mathbf{0.881}$(7.1 as Proper Integral)

= (3 – 9 + 24.3 – 52.071 + 91.125 – 134.202 + 170.334 – 189.800 + 188.405 – 168.573 + 137.266 – 102.543 + 70.755 – 45.355 + 27.146 – 15.237 + 8.051 – 4.019 + 1.901 – 0.854 + 0.366 – 0.149 + 0.058 – 0.022 + 0.008 – 0.003 + 0.001 – 0.000 + …….) – **0.881**

= 0.888 – **0.881**

≈ 0.007 (Approximated up to three decimal places)

7.3 Evaluation of $\int_3^4 e^{-x^2}\,dx$ by Bangabandhu function:

We know that the Bangabandhu function is

$B_{n*}(x) = \int e^{-x^n} dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!}$ + c, where $n \in$ N

Now by putting n = 2 in this function we get

$\int_3^4 e^{-x^2}\,dx = [\,B_{2*}(x)\,]_3^4 = B_{2*}(4) - B_{2*}(3)$

$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 4^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!} - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 3^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \frac{4^{2m-1}}{(2m-1).(m-1)!} - \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\ 3^{2m-1}}{(2m-1).(m-1)!}$

$=(\frac{4}{1.0!} - \frac{4^3}{3.1!} + \frac{4^5}{5.2!} - \frac{4^7}{7.3!} + \frac{4^9}{9.4!} - \frac{4^{11}}{11.5!} + \frac{4^{13}}{13.6!} - \frac{4^{15}}{15.7!} + \frac{4^{17}}{17.8!} - \frac{4^{19}}{19.9!} + \frac{4^{21}}{21.10!} - \frac{4^{23}}{23.11!} + \frac{4^{25}}{25.12!} - \frac{4^{27}}{27.13!} + \frac{4^{29}}{29.14!} - \frac{4^{31}}{31.15!} + \frac{4^{33}}{33.16!} - \frac{4^{35}}{35.17!} + \frac{4^{37}}{37.18!} - \frac{4^{39}}{39.19!} + \frac{4^{41}}{41.20!} - \frac{4^{43}}{43.21!} + \frac{4^{45}}{45.22!} - \frac{4^{47}}{47.23!} +$

$\frac{4^{49}}{49.24!} - \frac{4^{51}}{51.25!} + \frac{4^{53}}{53.26!} - \frac{4^{55}}{55.27!} + \frac{4^{57}}{57.28!} - \frac{4^{59}}{59.29!} + \frac{4^{61}}{61.30!} - \frac{4^{63}}{63.31!} + \frac{4^{65}}{65.32!} - \frac{4^{67}}{67.33!} + \frac{4^{69}}{69.34!} - \frac{4^{71}}{71.35!} + \frac{4^{73}}{73.36!} - \frac{4^{75}}{75.37!} + \frac{4^{77}}{77.38!} - \frac{4^{79}}{79.39!} + \frac{4^{81}}{81.40!} - \frac{4^{83}}{83.41!} + \frac{4^{85}}{85.42!} - \frac{4^{87}}{87.43!} + \frac{4^{89}}{89.44!} - \frac{4^{91}}{91.45!} + \frac{4^{93}}{93.46!} - \dots\dots\dots\dots\dots\dots\infty) - \mathbf{0.888}$ (7.2 as Proper Integral)

= (4 – 21.334 + 102.4 – 390.095 + 1213.630 – 3177.503 + 7169.750 – 14202.934 + 25064.001 – 39867.885 + 57713.510 – 76647.192 + 94020.555 – 107145.932 + 114007.494 – 113762.316 + 106867.630 – 94833.796 + 79740.129 – 63706.014 + 48478.723 – 35218.208 + 24474.876 – 16301.490 + 10424.082 – 6409.789 + 3795.637 – 2167.475 + 1195.099 – 637.014 + 328.602 – 164.217 + 79.582 – 37.433 + 17.105 – 7.599 + 3.285 – 1.383 + 0.567 – 0.227 + 0.088 – 0.034 + 0.013 – 0.005 + 0.002 – 0.001 + 0.000 - …….) – **0.888**

= 0.884 – **0.888**

≈ Modulus of (- 0.004) = 0.004 (Approximated up to three decimal places)

8. IN THE INFINITE INTERVAL [0, ∞) AND (- ∞, + ∞) AS IMPROPER INTEGRAL

8.1 Evaluation of $\int_0^{\infty} e^{-x^2}\,dx$

From subsubsection 6.2.2 and subsections 7.1, 7.2, and 7.3, we see that when x is increasing gradually, then the amount of extended area numerically increasing too much smaller, So we can say that if $x \to \infty$, then $\int_0^{\infty} e^{-x^2}\,dx$ can be approximated to $\int_0^{5} e^{-x^2}\,dx$, because e^{-x^2} is a continuous function everywhere and when $x \to 4, 5, 6, \dots\dots\infty$, then $e^{-x^2} \to 0$ (up to three decimal places)

We know that the Bangabandhu function is

$B_{n*}(x) = \int e^{-x^n} dx = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!}$ + c, where n ∈ N

Now by putting n = 2 in this function we get the Improper Integral,

$\int_0^{\infty} e^{-x^2}\,dx = \int_0^{5} e^{-x^2}\,dx = [\,B_{2*}(x)\,]_0^5 = B_{2*}(5) - B_{2*}(0)$

$= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\ 5^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!} - \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\ 0^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!}$

$= \sum_{m=1}^{\infty}(-1)^{m-1} \frac{5^{2m-1}}{(2m-1).(m-1)!} - 0 = \sum_{m=1}^{\infty}(-1)^{m-1} \frac{5^{2m-1}}{(2m-1).(m-1)!}$

$= \frac{5}{1.0!} - \frac{5^3}{3.1!} + \frac{5^5}{5.2!} - \frac{5^7}{7.3!} + \frac{5^9}{9.4!} - \frac{5^{11}}{11.5!} + \frac{5^{13}}{13.6!} - \frac{5^{15}}{15.7!} + \frac{5^{17}}{17.8!} - \frac{5^{19}}{19.9!} + \frac{5^{21}}{21.10!} - \frac{5^{23}}{23.11!} + \frac{5^{25}}{25.12!} - \frac{5^{27}}{27.13!} + \frac{5^{29}}{29.14!} - \frac{5^{31}}{31.15!} + \frac{5^{33}}{33.16!} - \frac{5^{35}}{35.17!} + \frac{5^{37}}{37.18!} - \frac{5^{39}}{39.19!} + \frac{5^{41}}{41.20!} - \frac{5^{43}}{43.21!} + \frac{5^{45}}{45.22!} - \frac{5^{47}}{47.23!} + \frac{5^{49}}{49.24!} - \frac{5^{51}}{51.25!} + \frac{5^{53}}{53.26!} - \frac{5^{55}}{55.27!} + \frac{5^{57}}{57.28!} - \frac{5^{59}}{59.29!} + \frac{5^{61}}{61.30!} - \frac{5^{63}}{63.31!} + \frac{5^{65}}{65.32!} - \frac{5^{67}}{67.33!} + \frac{5^{69}}{69.34!} - \frac{5^{71}}{71.35!} +$

$\frac{5^{73}}{73.36!} - \frac{5^{75}}{75.37!} + \frac{5^{77}}{77.38!} - \frac{5^{79}}{79.39!} + \frac{5^{81}}{81.40!} - \frac{5^{83}}{83.41!} + \frac{5^{85}}{85.42!} - \frac{5^{87}}{87.43!} + \frac{5^{89}}{89.44!} - \frac{5^{91}}{91.45!} + \frac{5^{93}}{93.46!} - \frac{5^{95}}{95.47!} + \frac{5^{97}}{97.48!} - \frac{5^{99}}{99.49!} + \frac{5^{101}}{101.50!} - \frac{5^{103}}{103.51!} + \frac{5^{105}}{105.52!} - \frac{5^{107}}{107.53!} + \frac{5^{109}}{109.54!} - \frac{5^{111}}{111.55!} + \frac{5^{113}}{113.56!} - \frac{5^{115}}{115.57!} + \frac{5^{117}}{117.58!} - \frac{5^{119}}{119.59!} + \frac{5^{121}}{121.60!} - \frac{5^{123}}{123.61!} + \frac{5^{125}}{125.62!} - \frac{5^{127}}{127.63!} + \frac{5^{129}}{129.64.!} - \frac{5^{131}}{131.65!} + \frac{5^{133}}{133.66!} - \frac{5^{135}}{135.67!} + \frac{5^{137}}{137.68!} - \frac{5^{139}}{139.69!} + \frac{5^{141}}{141.70!} - \ldots\ldots\ldots\ldots\ldots\ldots\infty$

= 5 – 41.667 + 312.5 – 1860.119 + 9042.245 – 36991.004 + 130417.001 – 403671.668 + 1113065.262 – 2766390.271 + 6257311.327 – 12984539.315 + 24887033.686 – 44314518.673 + 73675616.020 – 114870584.117 + 168607391.459 – 233783357.905 + 307147805.056 – 383416625.744 + 455891719.635 – 517485063.816 + 561915599.598 – 584787279.878 + 584290011.783 – 561376677.987 + 519416011.926 – 463451997.173 + 399277253.203 – 332536578.501 + 268028116.551 – 209289747.947 + 158476612.027 – 116474217.342 + 83160406.670 – 57727042.860 + 38989916.696 – 25642017.286 + 16431572.936 – 10266399.839 + 6258067.803 – 3723945.725 + 2164478.257 – 1229488.451 + 682874.714 – 371036.933 + 197313.937 – 102744.659 + 52409.486 – 26199.341 + 12840.271 – 6172.032 + 2910.803 – 1347.356 + 612.331 – 273.317 + 119.857 – 51.655 + 21.884 – 9.117 + 3.736 – 1.506 + 0.598 – 0.233 + 0.090 – 0.034 + 0.013 – 0.005 + 0.002 – 0.001 + 0.000 - …….

≈ 0.886 (Approximated up to three decimal places) $\approx \frac{\sqrt{\pi}}{2}$

8.2 Evaluation of Gaussian Integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Since the function e^{-x^2} is symmetric about the Y-axis, continuous and an even function everywhere in the Set of Real Numbers, then the Improper Integral,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2 \times \int_0^{\infty} e^{-x^2}\,dx$$
$$= 2 \times 0.886$$
$$= 1.772 \text{ (Approximated up to three decimal places)}$$
$$\approx \sqrt{\pi}$$

that is approximately matches with the Gaussian Integral [3] $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

8.3 Evaluation of $\int_0^{\infty} e^{-x^4}\,dx$

Similarly we can say that if $x \to \infty$, then $\int_0^{\infty} e^{-x^4}\,dx$ can be approximated to $\int_0^{2} e^{-x^4}\,dx$, because e^{-x^4} is a continuous function everywhere and when $x \to 2, 3, 4, \ldots\ldots\infty$, then $e^{-x^4} \to 0$ (up to three decimal places)

We know that the Bangabandhu function is

$B_{n*}(x) = \int e^{-x^n}\,dx = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c$, where $n \in \mathrm{N}$

Now by putting n = 4 in this function we get the Improper Integral,

$$\int_0^{\infty} e^{-x^4}\,dx = \int_0^{2} e^{-x^4}\,dx = [\,B_{4*}(x)\,]_0^2 = B_{4*}(2) - B_{4*}(0)$$
$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 2^{4.m-(4-1)}}{\{(4.m-(4-1)\}.(m-1)!} - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ 0^{4.m-(4-1)}}{\{(4.m-(4-1)\}.(m-1)!}$$
$$= \sum_{m=1}^{\infty}(-1)^{m-1} \frac{2^{4m-3}}{(4m-3).(m-1)!} - 0 = \sum_{m=1}^{\infty}(-1)^{m-1} \frac{2^{4m-3}}{(4m-3).(m-1)!}$$

$= \frac{2}{1.0!} - \frac{2^5}{5.1!} + \frac{2^9}{9.2!} - \frac{2^{13}}{13.3!} + \frac{2^{17}}{17.4!} - \frac{2^{21}}{21.5!} + \frac{2^{25}}{25.6!} - \frac{2^{29}}{29.7!} + \frac{2^{33}}{33.8!} - \frac{2^{37}}{37.9!} + \frac{2^{41}}{41.10!} - \frac{2^{45}}{45.11!} + \frac{2^{49}}{49.12!} - \frac{2^{53}}{53.13!} + \frac{2^{57}}{57.14!} - \frac{2^{61}}{61.15!} + \frac{2^{65}}{65.16!} - \frac{2^{69}}{69.17!} + \frac{2^{73}}{73.18!} - \frac{2^{77}}{77.19!} + \frac{2^{81}}{81.20!} - \frac{2^{85}}{85.21!} + \frac{2^{89}}{89.22!} - \frac{2^{93}}{93.23!} + \frac{2^{97}}{97.24!} - \frac{2^{101}}{101.25!} + \frac{2^{105}}{105.26!} - \frac{2^{109}}{109.27!} + \frac{2^{113}}{113.28!} - \frac{2^{117}}{117.29!} + \frac{2^{121}}{121.30!} - \frac{2^{125}}{125.31!} + \frac{2^{129}}{129.32.!} - \frac{2^{133}}{133.33!} + \frac{2^{137}}{137.34!} - \frac{2^{141}}{141.35!} + \frac{2^{145}}{145.36!} - \frac{2^{149}}{149.37!} + \frac{2^{153}}{153.38!} - \frac{2^{157}}{157.39!} + \frac{2^{161}}{161.40!} - \frac{2^{165}}{165.41!} + \frac{2^{169}}{169.42!} - \frac{2^{173}}{173.43!} + \frac{2^{177}}{177.44!} - \cdots \ldots \ldots \ldots \ldots \ldots \infty$

= 2 – 6.4 + 28.444 – 105.026 + 321.255 – 832.203+ 1864.135 – 3673.173 + 6455.879 – 10236.349 + 14780.289 – 19587.616 + 23984.835 – 27291.888 + 29001.906 – 28906.818 + 27127.937 – 24052.050 + 20208.115 – 16133.341 + 12269.306 – 8908.135 + 6187.469 – 4119.194 + 2632.887 – 1618.313 + 957.946 – 546.840 + 301.419 – 160.615 + 82.829 – 41.383 + 20.050 – 9.429 + 4.307 – 1.913 + 0.827 – 0.348 + 0.143 – 0.057 + 0.022 – 0.008 + 0.003 – 0.001 + 0.000 - …….

≈ 0.903 (Approximated up to three decimal places) that is approximately matches with [4, 6]

8.4 Evaluation of $\int_{-\infty}^{\infty} e^{-x^4}\, dx$

Since the function e^{-x^4} is symmetric about the Y-axis, continuous and an even function everywhere in the Set of Real Numbers, then the Improper Integral,

$\int_{-\infty}^{\infty} e^{-x^4}\, dx = 2 \times \int_{0}^{\infty} e^{-x^4}\, dx$

$= 2 \times 0.903$

$= 1.806$ (Approximated up to three decimal places)

8.5 Evaluation of $\int_{-\infty}^{0} e^{x}\, dx$

Similarly we can say that if $x \to -\infty$, then $\int_{-\infty}^{0} e^{x}\, dx$ can be approximated to $\int_{-8}^{0} e^{x}\, dx$, because e^{x} is a continuous function everywhere and when $x \to -8, -9, -10, \ldots \ldots -\infty$, then $e^{x} \to 0$ (up to three decimal places)

We know that the Bangabandhu function is

$B_n(x) = \int e^{x^n} dx = \sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c$, where $n \in \mathrm{N}$

Now by putting n = 1 in this function we get the Improper Integral,

$\int_{-\infty}^{0} e^{x}\, dx = \int_{-8}^{0} e^{x}\, dx = [\, B_1(x)\,]_{-8}^{0} = B_1(0) - B_1(-8)$

$= \sum_{m=1}^{\infty} \frac{0^{1.m-(1-1)}}{\{(1.m-(1-1)\}.(m-1)!} - \sum_{m=1}^{\infty} \frac{(-8)^{1.m-(1-1)}}{\{(1.m-(1-1)\}.(m-1)!}$

$= 0 - \sum_{m=1}^{\infty} \frac{(-8)^m}{m.(m-1)!} = -(-1)^m \sum_{m=1}^{\infty} \frac{8^m}{m.(m-1)!}$

$= \frac{8}{1.0!} - \frac{8^2}{2.1!} + \frac{8^3}{3.2!} - \frac{8^4}{4.3!} + \frac{8^5}{5.4!} - \frac{8^6}{6.5!} + \frac{8^7}{7.6!} - \frac{8^8}{8.7!} + \frac{8^9}{9.8!} - \frac{8^{10}}{10.9!} + \frac{8^{11}}{11.10!} - \frac{8^{12}}{12.11!} + \frac{8^{13}}{13.12!} - \frac{8^{14}}{14.13!} + \frac{8^{15}}{15.14!} - \frac{8^{16}}{16.15!} + \frac{8^{17}}{17.16!} - \frac{8^{18}}{18.17!} + \frac{8^{19}}{19.18!} - \frac{8^{20}}{20.19!} + \frac{8^{21}}{21.20!} - \frac{8^{22}}{22.21!} + \frac{8^{23}}{23.22!} - \frac{8^{24}}{24.23!} + \frac{8^{25}}{25.24!} - \frac{8^{26}}{26.25!} + \frac{8^{27}}{27.26!} - \cdots \ldots \ldots \ldots \ldots \infty$

= 8 – 32 + 85.333 – 170.667 + 273.067 – 364.089 + 416.102 – 416.102 + 369.868 – 295.894 + 215.196 – 143.464 + 88.286 – 50.449 + 26.906 – 13.453 + 6.331 – 2.814 + 1.185 – 0.474 + 0.181 – 0.066 + 0.023 – 0.008 + 0.002 – 0.001 + 0.000 - ………………….

≈ 0.999 ≈ 1 (Approximated up to three decimal places) that is approximately matches with [4, 6]

8.6 Evaluation of $\int_{-\infty}^{0} e^{x^3}\,dx$

Also similarly we can say that if $x \to -\infty$, then $\int_{-\infty}^{0} e^{x^3}\,dx$ can be approximated to $\int_{-2}^{0} e^{x^3}\,dx$, because e^{x^3} is a continuous function everywhere and when $x \to -2, -3, -4, \dots \dots -\infty$, then $e^{x^3} \to 0$ (up to three decimal places)

We know that the Bangabandhu function is

$B_n(x) = \int e^{x^n} dx = \sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c$, where $n \in N$

Now by putting n = 3 in this function we get the Improper Integral,

$$\int_{-\infty}^{0} e^{x^3}\,dx = \int_{-2}^{0} e^{x^3}\,dx = [\,B_3(x)\,]_{-2}^{0} = B_3(0) - B_3(-2)$$

$$= \sum_{m=1}^{\infty} \frac{0^{3.m-(3-1)}}{\{(3.m-(3-1)\}.(m-1)!} - \sum_{m=1}^{\infty} \frac{(-2)^{3.m-(3-1)}}{\{(3.m-(3-1)\}.(m-1)!}$$

$$= 0 - \sum_{m=1}^{\infty} \frac{(-1)^{3m-2}\,2^{3m-2}}{(3m-2).(m-1)!} = -(-1)^{3m-2} \sum_{m=1}^{\infty} \frac{2^{3m-2}}{(3m-2).(m-1)!}$$

$$= \frac{2}{1.0!} - \frac{2^4}{4.1!} + \frac{2^7}{7.2!} - \frac{2^{10}}{10.3!} + \frac{2^{13}}{13.4!} - \frac{2^{16}}{16.5!} + \frac{2^{19}}{19.6!} - \frac{2^{22}}{22.7!} + \frac{2^{25}}{25.8!} - \frac{2^{28}}{28.9!} + \frac{2^{31}}{31.10!} - \frac{2^{34}}{34.11!} + \frac{2^{37}}{37.12!} - \frac{2^{40}}{40.13!} + \frac{2^{43}}{43.14!} - \frac{2^{46}}{46.15!} + \frac{2^{49}}{49.16!} - \frac{2^{52}}{52.17!} + \frac{2^{55}}{55.18!} - \frac{2^{58}}{58.19!} + \frac{2^{61}}{61.20!} - \frac{2^{64}}{64.21!} + \frac{2^{67}}{67.22!} - \frac{2^{70}}{70.23!} + \frac{2^{73}}{73.24!} - \dots \dots \dots \dots \dots \infty$$

= 2 – 4 + 9.143 – 17.067 + 26.256 – 34.133 + 38.325 – 37.827 + 33.288 – 26.419 + 19.090 – 12.659 + 7.755 – 4.414 + 2.346 – 1.170 + 0.549 – 0.243 + 0.102 - 0.041 + 0.016 - 0.006 + 0.002 - 0.001 + 0.000 -…………..

≈ 0.892 (Approximated up to three decimal places) that is approximately matches with [4, 6]

8.7 Evaluation of $\int_{0}^{\infty} e^{x^2}\,dx$

We know that the Bangabandhu function is

$B_n(x) = \int e^{x^n} dx = \sum_{m=1}^{\infty} \frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!} + c$, where n ∈ N

Now by putting n = 2 in this function we get the Improper Integral,

$$\int_0^{\infty} e^{x^2}\,dx = \int_0^5 e^{x^2}\,dx + \int_5^{\infty} e^{x^2}\,dx = [\,B_2(x)\,]_0^5 + \int_5^{\infty} e^{x^2}\,dx$$

$$= B_2(5) - B_2(0) + \int_5^{\infty} e^{x^2}\,dx$$

$$= \sum_{m=1}^{\infty} \frac{5^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!} - \sum_{m=1}^{\infty} \frac{0^{2.m-(2-1)}}{\{(2.m-(2-1)\}.(m-1)!} + \int_5^{\infty} e^{x^2}\,dx$$

$$= \sum_{m=1}^{\infty} \frac{5^{2m-1}}{(2m-1).(m-1)!} - 0 + \int_5^{\infty} e^{x^2}\,dx = \sum_{m=1}^{\infty} \frac{5^{2m-1}}{(2m-1).(m-1)!} + \int_5^{\infty} e^{x^2}\,dx$$

$$= \frac{5}{1.0!} + \frac{5^3}{3.1!} + \frac{5^5}{5.2!} + \frac{5^7}{7.3!} + \frac{5^9}{9.4!} + \frac{5^{11}}{11.5!} + \frac{5^{13}}{13.6!} + \frac{5^{15}}{15.7!} + \frac{5^{17}}{17.8!} + \frac{5^{19}}{19.9!} + \frac{5^{21}}{21.10!} + \frac{5^{23}}{23.11!} + \frac{5^{25}}{25.12!} + \frac{5^{27}}{27.13!} + \frac{5^{29}}{29.14!} + \frac{5^{31}}{31.15!} + \frac{5^{33}}{33.16!} + \frac{5^{35}}{35.17!} + \frac{5^{37}}{37.18!} + \frac{5^{39}}{39.19!} + \frac{5^{41}}{41.20!} + \frac{5^{43}}{43.21!} + \frac{5^{45}}{45.22!} + \frac{5^{47}}{47.23!} + \frac{5^{49}}{49.24!} + \frac{5^{51}}{51.25!} + \frac{5^{53}}{53.26!} + \frac{5^{55}}{55.27!} + \frac{5^{57}}{57.28!} + \frac{5^{59}}{59.29!} + \frac{5^{61}}{61.30!} + \frac{5^{63}}{63.31!} + \frac{5^{65}}{65.32!} + \frac{5^{67}}{67.33!} + \frac{5^{69}}{69.34!} + \frac{5^{71}}{71.35!} + \frac{5^{73}}{73.36!} + \frac{5^{75}}{75.37!} + \frac{5^{77}}{77.38!} + \frac{5^{79}}{79.39!} + \frac{5^{81}}{81.40!} + \frac{5^{83}}{83.41!} + \frac{5^{85}}{85.42!} + \frac{5^{87}}{87.43!} + \frac{5^{89}}{89.44!} + \frac{5^{91}}{91.45!} + \frac{5^{93}}{93.46!} + \frac{5^{95}}{95.47!} + \frac{5^{97}}{97.48!} + \frac{5^{99}}{99.49!} + \frac{5^{101}}{101.50!} + \frac{5^{103}}{103.51!} + \frac{5^{105}}{105.52!} + \frac{5^{107}}{107.53!} + \frac{5^{109}}{109.54!} + \frac{5^{111}}{111.55!} + \frac{5^{113}}{113.56!} + \frac{5^{115}}{115.57!} + \frac{5^{117}}{117.58!} + \frac{5^{119}}{119.59!} + \frac{5^{121}}{121.60!} + \frac{5^{123}}{123.61!} + \frac{5^{125}}{125.62!} + \frac{5^{127}}{127.63!} + \frac{5^{129}}{129.64.!} + \frac{5^{131}}{131.65!} + \frac{5^{133}}{133.66!} + \frac{5^{135}}{135.67!} + \frac{5^{137}}{137.68!} + \frac{5^{139}}{139.69!} + \frac{5^{141}}{141.70!} + \dots \dots \dots \dots \dots \dots \infty + \int_5^{\infty} e^{x^2}\,dx$$

= (5 + 41.667 + 312.5 + 1860.119 + 9042.245 + 36991.004 + 130417.001 + 403671.668 + 1113065.262 + 2766390.271 + 6257311.327 + 12984539.315 + 24887033.686 + 44314518.673 + 73675616.020 + 114870584.117 + 168607391.459 + 233783357.905 + 307147805.056 + 383416625.744 + 455891719.635 + 517485063.816 + 561915599.598 + 584787279.878 + 584290011.783 + 561376677.987 + 519416011.926 + 463451997.173 + 399277253.203 + 332536578.501 + 268028116.551 + 209289747.947 + 158476612.027 + 116474217.342 + 83160406.670 + 57727042.860 + 38989916.696 + 25642017.286 + 16431572.936 + 10266399.839 + 6258067.803 + 3723945.725 + 2164478.257 + 1229488.451 + 682874.714 + 371036.933 + 197313.937 + 102744.659 + 52409.486 + 26199.341 + 12840.271 + 6172.032 + 2910.803 + 1347.356 + 612.331 + 273.317 + 119.857 + 51.655 + 21.884 + 9.117 + 3.736 + 1.506 + 0.598 + 0.233 + 0.090 + 0.034 + 0.013 + 0.005 + 0.002 + 0.001 + 0.000 + …….∞) + $\int_5^\infty e^{x^2}\,dx$

= $7.354\times10^9 + \int_5^\infty e^{x^2}\,dx$ [4, 6]

Analysis of the above integral result:
It indicates that if we work on the limit from 0 to 5, the integral becomes convergent. But if we increase the upper limit, then the result becomes larger that greater than any large number that we can imagine. So the final results will be infinity. Therefore the integral is divergent. i.e.,$\int_{-\infty}^{0} e^{x^2}\,dx$, $\int_0^\infty e^{x^2}\,dx$, and $\int_{-\infty}^{\infty} e^{x^2}\,dx$ are divergent.

9. DISCUSSION

The proposed functions i.e., Bangabandhu functions $B_n(x) = \int e^{x^n}dx = \sum_{m=1}^{\infty}\frac{x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!}$ + c, where $n \in \mathrm{N}$ and $B_{n*}(x) = \int e^{-x^n}dx = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\ x^{n.m-(n-1)}}{\{(n.m-(n-1)\}.(m-1)!}$ + c, where $n \in \mathrm{N}$ can be considered not the only the alternative of Gaussian Integral but also these functions are eligible for evaluating or computing $\int e^{x^2}dx$, $\int e^{x^3}dx$, $\int e^{x^4}dx$, $\int e^{-x^2}dx$, $\int e^{-x^3}dx$, $\int e^{-x^4}dx$, $\int_a^b e^{x^2}\,dx$, $\int_a^b e^{x^3}\,dx$, $\int_a^b e^{x^4}\,dx$, $\int_a^b e^{-x^2}\,dx$, $\int_a^b e^{-x^3}\,dx$, $\int_a^b e^{-x^4}\,dx$, ….,etc. without using erf(x), erfi(x), and erfc(x), and Gamma function. When calculating we just have to be careful so that no numerical value is omitted in the process of addition and subtraction. I think when calculated carefully it gives the gift of indefinite integration or anti-derivative function on the one hand and on the other hand, definite integrals can be solved successfully.
From subsections 8.1 to 8.7 and by the characteristics of the functions, we can be able to say that,

i) If $n \in \mathrm{N}$ is odd, then $\int_0^\infty e^{-x^n}\,dx$ converges because as x increases from zero, e^{-x^n} approaches zero, but $\int_{-\infty}^{0} e^{-x^n}\,dx$ and $\int_{-\infty}^{\infty} e^{-x^n}\,dx$ are divergent.

ii) If $n \in \mathrm{N}$ is even, then $\int_{-\infty}^{0} e^{-x^n}\,dx$, $\int_0^\infty e^{-x^n}\,dx$, and $\int_{-\infty}^{\infty} e^{-x^n}\,dx$ all are convergent because as x increases or decreases from zero, e^{-x^n} approaches zero.

iii) If $n \in \mathrm{N}$ is odd, then $\int_{-\infty}^{0} e^{x^n}\,dx$ converges because as x decreases from zero, e^{x^n} approaches zero, but $\int_0^\infty e^{x^n}\,dx$ and $\int_{-\infty}^{\infty} e^{x^n}\,dx$ are divergent.

iv) If $n \in \mathrm{N}$ is even, then $\int_{-\infty}^{0} e^{x^n}\,dx$, $\int_0^\infty e^{x^n}\,dx$, $\int_{-\infty}^{\infty} e^{x^n}\,dx$ all are divergent because as x increases or decreases from zero, e^{x^n} increases without limit.

10. CONCLUSION

For integration, there are many functions whose anti-derivatives or indefinite integrals can't express as an elementary function for the error functions. Of these functions, e^{-x^2} is the most familiar and important function, discussed in Statistical Mechanics, Normal Distribution, Quantum Mechanics, Mathematics (for evaluating an area), etc. In that case, we use the Gaussian Integral successfully, but we have no general formula for finding out $\int e^{x^n}dx,\ n \in$ N and $\int e^{-x^n}dx,\ n \in$ N. In this paper, the endeavor was to establish generalized formulae for solving them. Although the proposed Bangabandhu functions $\boldsymbol{B_n(x)}$ and $\boldsymbol{B_{n*}(x)}$ are not purely elementary (there are two discrete variables m, n other than x involved), these can create a rhythm to calculate the above-mention type of problems. Every formula has merits and demerits. For proposed method, it is suitable for finding out anti-derivative functions, and definite integrals in the interval [0, 1], but if the intervals are other finite intervals and infinite intervals, then it is a time-consuming matter. However, these calculations are not so critical and if we aware of while calculation or iteration, then we can get the desired result easily. Hopping that, mathematicians and statisticians will try to do a better job for the development of such work.

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