Nano Fuzzy Continuous Functions

PurvaRajwade*,RachnaNavalakhe,Vaibhav Jain*****

* Department of Applied Mathematics, IET, DAVV, Indore (M.P.), India
**Department of Applied Mathematics and Computational Science, SGSITS, Indore (M.P.), India
***Department of Computer Science, IET, DAVV, Indore (M.P.), India

Abstract-The objective of this paper is to propose a new class of functions called Nano fuzzy continuous functions which can be defined on Nano fuzzy topological spaces and to derive their characterizations in terms of Nano fuzzy closed sets, Nano fuzzy closure and Nano fuzzy interior. The concept of Nano fuzzy topological space with respect to the fuzzy subset X of the universe is used here which is defined in terms of Nano fuzzy lower and Nano fuzzy upper approximations of$X$. Approximation operators are useful to find the relation between rough set theory and topology. In this paper we have tried to fuzzify the continuous functions with crisp approximations. Also, we have tried to extend our work by defining Nano fuzzy open maps, Nano fuzzy closed maps and Nano fuzzy homeomorphism.

Index Terms- Nano fuzzy topological space, Nano fuzzy open set, Nano fuzzy closed set, Nano fuzzy interior, Nano fuzzy closure, Nano fuzzy continuous functions, Nano fuzzy open maps, Nano fuzzy closed maps, Nano fuzzy homeomorphisms.

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I.INTRODUCTION

Topology consists of continuous functions as one of the core concepts. Continuous functions can be understood with the logic that its outcomes are directly affected by its inputs where it does not matter how small change it is. Firstly, Nano topology is introduced by L. Thivagar [4] and to define such topological spaces, approximations and boundary region of a subset of a universe with equivalence relation on it were required. They have also defined Nano closed sets, Nano open sets, Nano interior and Nano closure. In 2013, L. Thivagar and C. Richards [3] defined Nano continuous functions, Nano open maps, Nano closed maps and Nano homeomorphisms. Later, R. Navalakhe and P. Rajwade [2] defined Nano fuzzy topology in the same manner by fuzzifying the concepts of Nano topology. In this paper we have introduced a class of functions on Nano fuzzy topological spaces called Nano fuzzy continuous functions and characterized it in terms of Nano fuzzy closed sets, Nano fuzzy closure and Nano fuzzy interior. We have given expansion to our work by defining Nano fuzzy open maps, Nano fuzzy closed maps and Nano fuzzy homeomorphismsand their representation in terms of Nano fuzzy closure and Nano fuzzy interior.

II. PRELIMINARIES

Definition 2.1 [1] Let U be the universe, R be an equivalence relation on$U$ and$\tau_R(X) = \{U, \phi, L_R(X), U_R(X),\ B_R(X)\}$, where $X \subseteq U$. Then if $\tau_R(X)$ satisfies the following axioms:

i. $U\ and\ \phi$ belongs to$\tau_R(X)$,

ii. $\tau_R(X)$ is closed for the union of the elements of any sub collection of $\tau_R(X)$,

iii. $\tau_R(X)$ is closed for the intersection of the elements of any finite sub collection of $\tau_R(X)$,

Then the structure $(U, \tau_R(X))$ is called the Nano topological space. The elements which belong to $\tau_R(X)$ are called Nano open sets. Those sets which are not Nano open are called Nano closed sets.

Definition 2.2 [1] If$(U, \tau_R(X))$ is a Nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then the Nano interior of A is defined as the union of all Nano open subsets of A and is denoted by $NInt\ (A)$. It is the largest Nano open subset of A, contained in A.

The Nano closure of A is defined as the intersection of all Nano closed sets containing A and it is the smallest Nano closed set, denoted by $NCl(A)$.

Remark 2.3 [1] If$\tau_R(X)$is the Nano topology on U with respect to X then the set $B = \{U, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

Definition 2.4 [3] Let$(U, \tau_R(X))$ and $(V, \tau_{R'}(Y))$ be Nano topological spaces. Then a mapping $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$is Nano continuous on U if the inverse image of every Nano open set in V is Nano open in$U$.

Definition 2.5 [2] Let X be the universe, R be a fuzzy equivalence relation on X and λ be any fuzzy subset of$X$. Then $\tau_R(\lambda) = \{0_\lambda, 1_\lambda, \underline{R}(\lambda), \overline{R}(\lambda), Bd(\lambda)\}$satisfies the following axioms:

i. $0_\lambda, 1_\lambda \in \tau_{(R)}(\lambda)$ where $0_\lambda: \lambda \to I$ denotes the null fuzzy sets and $1_\lambda: \lambda \to I$ denotes the whole fuzzy set.
ii. Arbitrary union of members of $\tau_{(R)}(\lambda)$is a member of $\tau_{(R)}(\lambda)$.
iii. Finite intersection of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.

Then the pair $(X, \tau_{(R)}(\lambda))$ is called Nano fuzzy topological space and $\tau_{(R)}(\lambda)$ is a Nano fuzzy topology with respect toλ. Elements of the Nano fuzzy topology are called Nano fuzzy open sets and elements of $[\tau_{(R)}(\lambda)]^C$ are called Nano fuzzy closed sets.

Definition 2.6 [2] The basis for the Nano fuzzy topology $\tau_{(R)}(\lambda)$ with respect to $\lambda$is given by

$$B = \{1_\lambda, \underline{R}(\lambda)(x), B(\lambda)(x)\}$$

Definition 2.7 [2] Let$(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space with respect toλ where $\lambda \leq X$ and if $\mu \leq X$ then the Nano fuzzy interior of μ is defined as union of all Nano fuzzy open subsets of μ and it is denoted by $NFInt(\mu)$. That is, it is the largest Nano fuzzy open subset of μ.

Similarly, the Nano fuzzy closure of μ is defined as the intersection of all Nano fuzzy closed sets containing μ. It is denoted by $NFCl(\mu)$ and it is the smallest Nano fuzzy closed set containing μ.

III. NANO FUZZY CONTINUITY

Definition 3.1 Let $\left(X, \tau_{(R)}(\lambda)\right)$ and $\left(Y, \tau_{(R')}(\mu)\right)$ be Nano fuzzy topological spaces. Then a mapping $f: \left(X, \tau_{(R)}(\lambda)\right) \to \left(Y, \tau_{(R')}(\mu)\right)$ is Nano fuzzy continuous on X if the inverse image of every Nano fuzzy open set in Y is Nano fuzzy open in$X$.

Theorem 3.2 A function$f: \left(X, \tau_{(R)}(\lambda)\right) \to \left(Y, \tau_{(R')}(\mu)\right)$ is Nano fuzzy continuous if and only if the inverse image of every Nano fuzzy closed set in Y is Nano fuzzy closed in X.

Proof: Let f be Nano fuzzy continuous and ν be Nano fuzzy closed in Y. That is $1_Y - \nu$ is Nano fuzzy open in Y. Since f is Nano fuzzy continuous, $f^{-1}(1_Y - \nu)$is Nano fuzzy open in X. That is, $X - f^{-1}(\nu)$ is Nano fuzzy open in X. Therefore,$f^{-1}(\nu)$ is Nano fuzzy closed in X. Thus, the inverse image of every Nano fuzzy closed set in Y is Nano fuzzy closed in X, if f is Nano fuzzy continuous on X. Conversely, let the inverse image of every Nano fuzzy closed set be Nano fuzzy closed. Let α be Nano fuzzy open in Y. Then $1_Y - \alpha$ is Nano fuzzy closed in Y. Then, $f^{-1}(1_Y - \alpha)$ is Nano fuzzy closed in X. That is, $X - f^{-1}(\alpha)$is Nano fuzzy closed in X. Therefore,$f^{-1}(\alpha)$ is Nano fuzzy open in X. Thus, the inverse image of every Nano fuzzy open set in Y is Nano fuzzy open in X. That is, f is Nano fuzzy continuous on X.

Theorem 3.3 A function $f: \left(X, \tau_{(R)}(\lambda)\right) \to \left(Y, \tau_{(R')}(\mu)\right)$ is Nano fuzzy continuous if and only if $f\left(NfCl(\beta)\right) \leq NfCl\left(f(\beta)\right)$ for every fuzzy subset β of X.

Proof: Let f be Nano fuzzy continuous function and $\beta \leq X$. Then $f(\beta) \leq Y. NfCl\left(f(\beta)\right)$ is Nano fuzzy closed in Y. Since f is Nano fuzzy continuous, $f^{-1}\left(NfCl\left(f(\beta)\right)\right)$ is Nano fuzzy closed in X.
Since $f(\beta) \leq NfCl\left(f(\beta)\right)$, $\beta \leq f^{-1}\left(NfCl\left(f(\beta)\right)\right)$. Thus, $f^{-1}\left(NfCl\left(f(\beta)\right)\right)$ is a Nano fuzzy closed set containing β. But, $NfCl(\beta)$ is the smallest Nano fuzzy closed containing β. Therefore, $NfCl(\beta) \leq f^{-1}\left(NfCl(\beta)\right)$.
That is, $f\left(NfCl(\beta)\right) \leq NfCl(\beta)$.
Conversely, let $f\left(NfCl(\beta)\right) \leq NfCl(\beta)$ for every fuzzy subset $\beta$of X. If α is Nano fuzzy closed in$Y$, since$f^{-1}(\alpha) \leq X, f\left(NfCl\left(f^{-1}(\alpha)\right)\right) \leq NfCl\left(f\left(f^{-1}(\alpha)\right)\right) = NfCl(\alpha)$.
That is, $NfCl\left(f^{-1}(\alpha)\right) \leq f^{-1}\left(NfCl(\alpha)\right) = f^{-1}(\alpha)$, since α is Nano fuzzy closed.Thus, $NfCl\left(f^{-1}(\alpha)\right) \leq f^{-1}(\alpha)$.But $f^{-1}(\alpha) \leq NfCl\left(f^{-1}(\alpha)\right)$. Therefore, $NfCl\left(f^{-1}(\alpha)\right) = f^{-1}(\alpha)$. So $f^{-1}(\alpha)$is Nano fuzzy closed in X for every Nano fuzzy closed set αin Y. That is, f is Nano fuzzy continuous.

Theorem 3.4 A function $f: \left(X, \tau_{(R)}(\lambda)\right) \to \left(Y, \tau_{(R')}(\mu)\right)$ is Nano fuzzy continuous if and only if $NfCl\left(f^{-1}(\alpha)\right) \leq f^{-1}\left(NfCl(\alpha)\right)$ for every fuzzy subset α of Y.

Proof: Let f be Nano fuzzy continuous function and $\alpha \leq X$, $NfCl(\alpha)$ is Nano fuzzy closed in Y and hence $f^{-1}\left(NfCl(\alpha)\right)$ is Nano fuzzy closed in$X$. Therefore, $NfCl\left[f^{-1}\left(NfCl(\alpha)\right)\right] = f^{-1}\left(NfCl(\alpha)\right)$. Since $\alpha \leq NfCl(\alpha)$, $f^{-1}(\alpha) \leq f^{-1}\left(NfCl(\alpha)\right)$. Therefore,
$NfCl\left(f^{-1}(\alpha)\right) \leq NfCl\left(f^{-1}\left(NfCl(\alpha)\right)\right) = f^{-1}\left(NfCl(\alpha)\right)$.
That is, $NfCl\left(f^{-1}(\alpha)\right) \leq f^{-1}\left(NfCl(\alpha)\right)$. Conversely, let $NfCl\left(f^{-1}(\alpha)\right) \leq f^{-1}\left(NfCl(\alpha)\right)$ for every $\alpha \leq X$. Let α be Nano fuzzy closed in Y. Then $NfCl(\alpha) = \alpha$.
By assumption, $NfCl\left(f^{-1}(\alpha)\right) \leq f^{-1}\left(NfCl(\alpha)\right) = f^{-1}(\alpha)$.
Thus, $NfCl\left(f^{-1}(\alpha)\right) \leq f^{-1}(\alpha)$. But, $f^{-1}(\alpha) \leq NfCl\left(f^{-1}(\alpha)\right)$. So, $NfCl\left(f^{-1}(\alpha)\right) = f^{-1}(\alpha)$. That is, $f^{-1}(\alpha)$ is Nano fuzzy closed in X for every Nano fuzzy closed set αin Y. Therefore, f is Nano fuzzy continuous on X.

The following theorem is an attempt toestablishcriteriafor Nano fuzzy continuous functions in terms of inverse image of Nano fuzzy interior of a subset of Y.

Theorem 3.5 A function $f: \left(X, \tau_{(R)}(\lambda)\right) \to \left(Y, \tau_{(R')}(\mu)\right)$ is Nano fuzzy continuous on X if and only if $f^{-1}\left(NfInt(\alpha)\right) \leq NfInt\left(f^{-1}(\alpha)\right)$ for every fuzzy subset α of Y.

Proof: Let f be Nano fuzzy continuous function and $\alpha \leq X$.

Then $NfInt\,(\alpha)$ is Nano fuzzy open in Y. Therefore, $f^{-1}(NfInt(\alpha))$ isNano fuzzy open in X.
That is, $f^{-1}(NfInt(\alpha)) = NfInt[f^{-1}(NfInt(\alpha))]$. Also, $NfInt(\alpha) \leq \alpha$ implies that $f^{-1}(NfInt(\alpha)) \leq f^{-1}(\alpha)$. So, $NfInt[f^{-1}(NfInt(\alpha))] \leq NfInt(f^{-1}(\alpha))$. That is, $f^{-1}(NfInt(\alpha)) \leq NfInt(f^{-1}(\alpha))$. Conversely, let $f^{-1}(NfInt(\alpha)) \leq NfInt(f^{-1}(\alpha))$for every fuzzy subset α of Y. If α is Nano fuzzy open in Y, $NfInt(\alpha) = \alpha$. So, $f^{-1}(\alpha) \leq NfInt(f^{-1}(\alpha))$. But, $NfInt(f^{-1}(\alpha)) \leq f^{-1}(\alpha)$. Therefore, $f^{-1}(\alpha) = NfInt(f^{-1}(\alpha))$. Thus, $f^{-1}(\alpha)$ is Nano fuzzy open in X for every Nano fuzzy open set αin Y. Therefore, f is Nano fuzzy continuous.

Theorem 3.6 If $(X, \tau_{(R)}(\lambda))$ and $(Y, \tau_{(R')}(\mu))$ are Nano fuzzy topological spaces with respect to $\lambda \leq X$ and $\mu \leq Y$ respectively, then for any function $f: X \to Y$, the following are equivalent:

- *i.* f is Nano fuzzy continuous,
- *ii.* The inverse image of every Nano fuzzy closed set in Y is Nano fuzzy closed in X.
- *iii.* $f(NfCl(\beta)) \leq NfCl(f(\beta))$for every fuzzy subset β of X.
- *iv.* $NfCl(f^{-1}(\alpha)) \leq f^{-1}(NfCl(\alpha))$ for every fuzzy subset α of Y.
- *v.* $f^{-1}(NfInt(\alpha)) \leq NfInt(f^{-1}(\alpha))$ for every fuzzy subset α of Y.

Proof: Proof of the theorem follows from theorems 3.2 to 3.5.

Definition 3.7 A fuzzy subset α of Nano fuzzy topological space $(X, \tau_{(R)}(\lambda))$ is said to be Nano fuzzy dense if $NfCl(\alpha) = X$.

Theorem 3.8 Let $f: (X, \tau_{(R)}(\lambda)) \to (Y, \tau_{(R')}(\mu))$be an onto, Nano fuzzy continuous function. If α is Nano fuzzy dense in X, then $f(\alpha)$ is Nano fuzzy dense in Y.

Proof: Since α is Nano fuzzy dense in X, $NfCl(\alpha) = X$. Then $f(NfCl(\alpha)) = f(X) = Y$, since f is onto. Since f is Nano fuzzy continuous on X, $f(NfCl(\alpha)) \leq NfCl(f(\alpha))$. Therefore, $Y \leq NfCl(f(\alpha))$. But $NfCl(f(\alpha)) \leq Y$. Therefore,$Y = NfCl(f(\alpha))$. That is, $f(\alpha)$ is Nano fuzzy dense in Y. Thus, a Nano fuzzy continuous function maps Nano fuzzy dense sets into Nano fuzzy dense sets, provided it is onto.

IV. NANO FUZZY OPEN MAPS, NANO FUZZY CLOSED SETS AND NANO FUZZY HOMEOMORPHISMS

Definition 4.1 A function$f: (X, \tau_R(\lambda)) \to (Y, \tau_{R'}(\mu))$is a Nano fuzzy open map if the image of every Nano fuzzy open set in X is Nano fuzzy open in Y. The mapping f is said to be a Nano fuzzy closed map if the image of every Nano fuzzy closed set in X is Nano fuzzy closed in Y.

Theorem 4.2 A mapping$f: (X, \tau_{(R)}(\lambda)) \to (Y, \tau_{(R')}(\mu))$ is Nano fuzzy closed map if and only if $NfCl(f(\alpha)) \leq f(NfCl(\alpha))$, for every fuzzy subset α of X.

Proof: If f is Nano fuzzy closed, $f(NfCl(\alpha))$ is Nano fuzzy closed in$Y$, since $NfCl(\alpha)$ is Nano fuzzy closed in X.
Since $\alpha \leq NfCl(\alpha)$, $f(\alpha) \leq f(NfCl(\alpha))$. Thus $f(NfCl(\alpha))$ is Nano fuzzy closed set containing $f(\alpha)$. Therefore, $NfCl(f(\alpha)) \leq f(NfCl(\alpha))$. Conversely, if $NfCl(f(\alpha)) \leq f(NfCl(\alpha))$, for every fuzzy subset α of Y and if β is Nano fuzzy closed in X, then $NfCl(\alpha) = \alpha$ and hence $f(\alpha) \leq NfCl(f(\alpha)) \leq f(NfCl(\alpha)) = f(\alpha)$. Thus, $f(\alpha) = NfCl(f(\alpha))$. That is, $f(\alpha)$ is Nano fuzzy closed in Y. Therefore, f is Nano fuzzy closed map.

Theorem 4.3 A mapping$f: (X, \tau_{(R)}(\lambda)) \to (Y, \tau_{(R')}(\mu))$ is Nano fuzzy open map if and only if $f(NfInt(\alpha)) \leq NfInt(f(\alpha))$, for every fuzzy subset α of X.

Proof: The proof is similar to the proof of theorem 4.2.

Definition 4.4 A function $f: (X, \tau_R(\lambda)) \to (Y, \tau_{R'}(\mu))$ is said to be a Nano fuzzy homeomorphism if

- *i.* f is 1-1 and onto
- *ii.* f is Nano fuzzy continuous
- *iii.* f is Nano fuzzy open.

Theorem 4.5 Let $f: (X, \tau_R(\lambda)) \to (Y, \tau_{R'}(\mu))$be a one-one onto mapping. Then f is a Nano fuzzy homeomorphism if and only if f is Nano fuzzy closed and Nano fuzzy continuous.

Proof: Let f be a Nano fuzzy homeomorphism. Then f is Nano fuzzy continuous. Let α be an arbitrary Nano fuzzy closed set in X. Then $X - \alpha$ is Nano fuzzy open. Since f is Nano fuzzy open map, $f(X - \alpha)$ is Nano fuzzy open in Y. That is, $Y - f(\alpha)$ is Nano fuzzy open in Y. Therefore, $f(\alpha)$ is Nano fuzzy closed in Y. Thus, the image of every Nano fuzzy closed set in X is Nano fuzzy closed in Y. That is, f is Nano fuzzy closed. Conversely, let f be Nano fuzzy closed an continuous map. Let β be Nano fuzzy open set in X. Then $X - \beta$ is Nano fuzzy closed in X. Since f is Nano fuzzy closed map, $f(X - \beta) = Y - f(\beta)$ is Nano fuzzy closed in Y. Therefore, $f(\beta)$ is Nano fuzzy open in Y. Thus, f is Nano fuzzy open and hence is a Nano fuzzy homeomorphism.

Theorem 4.6 A one-one map$f: (X, \tau_R(\lambda)) \to (Y, \tau_{R'}(\mu))$ is a Nano fuzzy homeomorphism if and only if $NfCl(f(\alpha)) = f(NfCl(\alpha))$, for every fuzzy subset α of X.

Proof: If f is a Nano fuzzy homeomorphism, f is Nano fuzzy continuous and Nano fuzzy closed. If $\alpha \leq X$, $f\big(NfCl(\alpha)\big) \leq NfCl\big(f(\alpha)\big)$, since f is Nano fuzzy continuous. Since $NfCl(\alpha)$ is Nano fuzzy closed in X and $f$is Nano fuzzy closed, $f\big(NfCl(\alpha)\big)$is Nano fuzzy closed in Y, $NfCl\Big(f\big(NfCl(\alpha)\big)\Big) = f\big(NfCl(\alpha)\big)$. Since $\alpha \leq NfCl(\alpha)$, $f(\alpha) \leq f\big(NfCl(\alpha)\big)$ and hence $NfCl\big(f(\alpha)\big) \leq NfCl\Big(f\big(NfCl(\alpha)\big)\Big) = f\big(NfCl(\alpha)\big)$.
Therefore, $NfCl\big(f(\alpha)\big) \leq f\big(NfCl(\alpha)\big)$. Thus, $NfCl\big(f(\alpha)\big) = f\big(NfCl(\alpha)\big)$.
Conversely, if $NfCl\big(f(\alpha)\big) = f\big(NfCl(\alpha)\big)$, for every fuzzy subset α of X, then f is Nano fuzzy continuous.If α is Nano fuzzy closed in X, $NfCl(\alpha) = \alpha$ which implies $f\big(NfCl(\alpha)\big) = f(\alpha)$. Therefore, $NfCl\big(f(\alpha)\big) = f(\alpha)$. Thus, $f(\alpha)$ is Nano fuzzy closed in Y, for every Nano fuzzy closed set αin X. That is f is Nano fuzzy closed map. Also f is Nano fuzzy continuous. Thus, f is Nano fuzzy homeomorphism.

V. CONCLUSION

The theory of Nano continuous functions has a variety of applications in real life such as cost cutting in day to day life applications, growth of a plant over time, depreciation of machine and temperature at various times of the day. In this paper we have defined Nano fuzzy continuity and Nano fuzzy homeomorphisms Thus, Nano fuzzy continuous functions, in near future, can be helpful and applied to more day-to-day situations as fuzzfication of concepts provides good outcomes even when the data is scattered and insufficient.

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AUTHORS

First Author – PurvaRajwade, Research Scholar, Institute of Engineering and Technology, DAVV, Indore (M.P.), India. Email: rajwadepurva@gmail.com

Second Author – Dr. RachnaNavalakhe, M.Phil., Ph.D., Assistant Professor, ShriGovindramSeksaria Institute of Technology and Science, Indore (M.P.), India.
Email: sgsits.rachna@gmail.com

Third Author – Dr. Vaibhav Jain, B.Tech., Ph.D., Assistant Professor, Institute of Engineering and Technology, DAVV, Indore (M.P.), India.
Email: vjain@ietdavv.edu.in

Correspondence Author – PurvaRajwade
Email: rajwadepurva@gmail.com
Contact number: +91-8962307747