p-Order Prime Graph

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Abstract

Let G be a group with identity e. Let p be a prime number. The p-Order Prime graph graph $\Gamma_{pop}(G)$ of G is a graph with $V(\Gamma_{pop}(G)) = G - e$ and two distinct vertices x and y are adjacent in $\Gamma_{pop}(G)$ if and only if $GCD(O(x), O(y)) = p$. In this paper, we want to explore how the group theoretical properties of G can effect on the graph theoretical properties of $\Gamma_{pop}(G)$. Some characterizations for fundamental properties of $\Gamma_{pop}(G)$ have also been obtained.

Key words: p-Order Prime graph, finite group, self inverse element.

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1 Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and thereby investigating algebraic properties of the ring or group using the associated graph, for instance, see [1, 2, 3, 7, 8, 9, 10, 11, 12]. In the present article, to any group G, we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. Before starting, let us introduce some necessary notation and definitions.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph $\Gamma = (V, E)$, V denote the set of all vertices and E denote the set of all edges in Γ. The degree $deg_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $deg_\Gamma(v)$ simply by $deg(v)$. The order of Γ is defined $|V(\Gamma)|$ and its maximum and its minimum degrees will be denoted, respectively, by $\Delta(\Gamma)$ and $\delta(\Gamma)$. A graph Γ is regular if the degrees of all vertices of Γ are the same. A vertex of degree 0 is known as an *isolated vertex* of Γ. A graph Ω is called a subgraph of Γ if $V(\Omega) \subseteq V(\Gamma)$, $E(\Omega) \subseteq E(\Gamma)$. Let $\Gamma = (V, E)$ be a graph and let $S \subseteq V$. A simple graph Γ is said to be *complete* if every pair of distinct vertices of Γ are adjacent in Γ. A graph Γ is said to be *connected* if every pair of distinct vertices of Γ are connected by a path in Γ. An *Eulerian* graph has an *Eulerian trail*, a closed trail containing all vertices and edges. The *Union* of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph $\Gamma = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The *join* of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph denoted by $\Gamma_1 + \Gamma_2 = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{$Edges joining every vertex of V_1 with every vertex of $V_2\}$.

Let G be a group with identity e. The order of the group G is the number of elements in G and is denoted by $O(G)$. The order of an element a in a group G is the smallest positive integer k such that $a^k = e$. If no such integer exists, we say a has infinite order. The order of an element a is denoted $O(a)$. Let p be a prime number. A group G with $O(G) = p^k$ for some $k \in \mathbb{Z}^+$, is called a *p-group*.

Theorem 1.1 *[5](Lagrange) If G is a finite group and H is a subgroup of G then $o(H)$ divides $o(G)$.*

Theorem 1.2 *[5](Sylow) Let p be a prime and m, a positive integer such that p^m divides $O(G)$. Then there exists a subgroup H of G such that $O(H) = p^m$. If $p^m | O(G)$ and $p^{m+1} \nmid O(G)$, then G has a subgroup of order p^m.*

Theorem 1.3 *[5](Sylow) The number of p-Sylow subgroups in G, for a given prime, is of the form $1 + kp$. In particular, this number is a divisor of $O(G)$.*

Theorem 1.4 *[6] If G is a group of order pq, where p and q are primes, $p < q$, and p does not divide $q - 1$, then G is cyclic. In particular, G is isomorphic to $\mathbb{Z}_{pq}$.*

2 Preparation of Manuscript

In this section, we observe certain basic properties of p-order prime graphs.

Proposition 2.1 *Let G be a finite group with identity e.Let p be a prime number. Let $x \in G$ be an element. x and x^{-1} are non adjacent in $\Gamma_{pop}(G)$ if and only if $O(x) \neq p$.*

Proof. Let $x \in G$ be an element such that $O(x) \neq p$. Clearly $O(x)$ and $O(x^{-1})$ eaual and not equal to p. Therefore $GCD(O(x), O(x^{-1})) \neq p$. Hence x and x^{-1} are non adjacent in $\Gamma_{pop}(G)$. Conversely assume that x and x^{-1} are non adjacent in $\Gamma_{pop}(G)$. Suppose $O(x) = p$, then $O(x) = O(x^{-1}) = p$ and $GCD(O(x), O(x^{-1})) = p$. Therefore x and x^{-1} are adjacent in $\Gamma_{pop}(G)$, which is a condradiction. Hence $O(x) \neq p$.

Theorem 2.2 *For any finite group G, A vertex is isolated in $\Gamma_{pop}(G)$ if and only if p* *does not divide $O(x)$, where p is a prime number.*

Proof. Let G be a finite group and x be an element such that p does not divide $O(x)$. Clearly x is an isolated vertex of $\Gamma_{pop}(G)$. Conversely, assume that x is an isolated vertex of $\Gamma_{pop}(G)$. Suppose p divides $O(x)$. Then $p|O(G)$ and by Cauchy's theorem G has an element y of order p. Clearly x and y are adjacnet in $\Gamma_{pop}(G)$, which is a contradiction. Hence p does not divide $O(x)$.

Theorem 2.3 *Let G be a finite group. $\Gamma_{pop}(G)$ is a null graph if and only if G p does not divide $O(G)$, where p is a prime number.*

Proof. Let G be a finite group. Assume p does not divide $O(G)$. Clearly p does not divide order of any element in G and hence $\Gamma_{pop}(G)$ is null graph. Conversely assume that $\Gamma_{pop}(G)$ is null graph. Therefore each vertex of $\Gamma_{pop}(G)$ is isolated. By Theorem 2.2, p does not divide order of any element in G and hence p does not divide $O(G)$.

Proposition 2.4 *Let G be a finite group and q be number of edges in $\Gamma_{pop}(G)$. Let p be a prime number. Then $q \leq \frac{m(m-1)}{2}$ where m be number of mutiple of p order elements in G. Moreover, this bound is sharp.*

Proof. Let G be a finite group of order n. Let V_1 and V_2 be two partition of the vertex set of $\Gamma_{pop}(G)$ such that V_1 be the set of all mulitiple of p order elements of the group G and V_2 be the set of remaining elements of the group G. Let $|V_1| = m$ and $|V_2| = l$. By Theorem 2.2, the vertices of V_2 are isolated in $\Gamma_{pop}(G)$. Therefore the edges are joining two vertices from V_1 only. There are m vertices in V_1, the maximum number of edges in $\Gamma_{pop}(G)$ is $\frac{m(m-1)}{2}$. Moreover, for the group $\mathbb{Z}_p \times \mathbb{Z}_p$, $\Gamma_{pop}(\mathbb{Z}_p \times \mathbb{Z}_p) \cong K_{p-1}$ and for this graph the bound is sharp.

We now characterize the groups G for which the associated graph $\Gamma_{pop}(G)$ attains this bound.

Theorem 2.5 *Let G be a finite group and q be number of edges in $\Gamma_{pop}(G)$.Let p be a prime number. Then $q = \frac{m(m-1)}{2}$ where m be number of multiple of p order elements in G if and only G has exactly m elements of order p.*

Proof. Assume that $\Gamma_{pop}(G)$ is a graph with $\frac{m(m-1)}{2}$ edges where m be number of multiple of p order elements in G. Let V_1 be the set of multiple of p order elements in G. Therefore $|V_1| = m$. By assumption, the graph induced by the set V_1 is isomorpic to K_m. We have to prove every elements in V_1 is of order p. Suppose $x \in V_1$ is of order not equal to p then x and x^{-1} have same order and hence they are not adjacent in $\Gamma_{pop}(G)$, which is a contradiction. Therefore all the elements in V_1 is of order p. Convrsely assume that G has exactly m elements of order p. Since m is number of multiple of order p elements, G has m elements of order p and remaning elements of order not a multiple of p. Therefore all these m elements are adjacent each other. Hence the result follows.

Theorem 2.6 *Let G be a finite group of order n. Then $\Gamma_{pop}(G)$ is a complete graph if and only if $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \mathbb{Z}_p$.*

Proof. Let G be a group of order n.

Assume that $\Gamma_{pop}(G)$ is a complete graph. Suppose G has an element of order which is not a multiple of p, say x, then by Propostion 2.1 x and x^{-1} are non adjacent in $\Gamma_{pop}(G)$, which is a contradiction. Conversely, assume that $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \dots \mathbb{Z}_p$. Clearly every elements other than identity are of order p. Hence $\Gamma_{pop}(G)$ is complete.

3 p-Order prime graph of the Groups has orders pq and p^n.

In this section, we study about the p-Order prime graph of groups of orders pq and p^{n} for some prime numbers p.

Theorem 3.1 *Let G be a group of order pq, where $p < q$ and p and q are prime. Then $\Gamma_{pop}(G) \cong (K_{p-1} + \overline{K_r}) \cup \overline{K_{q-1}}$ where $r = (p-1)(q-1)$, if G is cyclic and $\Gamma_{pop}(G) \cong K_{q(p-1)} \cup \overline{K_{q-1}}$, if G is non cyclic.*

Proof. Case(i). Let G be a cyclic group. In this case G has an unique p-Sylow subgroup and an unique q-Sylow subgroup. Therefore $p-1$ elements have order p, $q-1$ elements have order q and $r = (p-1)(q-1)$ elements have order pq. Clearly no elements of order q are adjacent, no elements of order pq are adjacent and the element of order p is adjacent to all the elements of order p and pq. Hence $\Gamma_{pop}(G) \cong (K_{p-1} + \overline{K_r}) \cup \overline{K_{q-1}}$.

Case(ii). Let G be a non-cyclic group. In this case G has qp-Sylow subgroups and a unique q-Sylow subgroup. Therefore G has $q(p-1)$ elements of order p and $q-1$ elements of order q. Hence $\Gamma_{pop}(G) \cong K_{q(p-1)} \cup \overline{K_{q-1}}$.

Theorem 3.2 *Let G be a group of order p^2. Then $\Gamma_{pop}(G) \cong K_{p^2-1}$ or $\Gamma_{pop}(G) \cong K_{p-1} + \overline{K_{p(p-1)}}$.*

Proof. Let G be a group of order p^2. Clealy G is abelian. Therefore $G \cong \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence $\Gamma_{pop}(G) \cong K_{p^2-1}$ or $\Gamma_{pop}(G) \cong K_{p-1} + \overline{K_{p(p-1)}}$.

Theorem 3.3 *Let G be a abelian group of order p^n. G has unique p-sylow subgroup if and only if $\Gamma_{pop}(G) \cong K_{p-1} + \overline{K_{p(p^{n-1}-1)}}$.*

Proof. Let G be a abelian group of order p^n. Assume that G has unique p-sylow subgroup. Therefore only $p-1$ elements have order p and remaing non identity elements have order power of p. Therefore In $\Gamma_{pop}(G)$ the elements of order p is adjacent to all other elements. and no elements of order other than p are non adjacent. Hence $\Gamma_{pop}(G) \cong K_{p-1} + \overline{K_{p(p^{n-1}-1)}}$. Conversely assume that $\Gamma_{pop}(G) \cong K_{p-1} + \overline{K_{p(p^{n-1}-1)}}$. Since G is a group of order p^n, G has element of order p and multiple of p. Suppose G has more than p-1 elements of order p. Note that the elements of order p are adjacent to all other order elements, $\Gamma_{pop}(G) \cong K_{p-1} + \overline{K_{p(p^{n-1}-1)}}$. Therefore G has an unique p sylow subgroup.

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