MEETING COMBINATIONS AND INTERVAL GRAPHS

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***Abstract*- This research was conducted to develop the important concepts on meeting combinations and interval graphs. It specifically aimed to (1) examine the properties of interval graphs; (2) determine the number of different arrangements for n segments of distinct identities; (3) determine the number of "meeting structures" or intersection for n segments and (4) establish the matrix representation and characteristic polynomial of interval graph.**

The results of the study revealed that, the vertices of an interval graph G can be ordered such that vi is connected to vk implies vj is connected to vk whenever i < j < k. The number of different arrangements at the endpoints for n segments is n!, which indicates that some of them have the same "meeting structure" or intersection denoting the number of distinct "meeting structure" for n segments by M(n), it was found out that for every value of M(n), there is a unique "meeting structure" or intersection for a given set or family of interval segments, for n ≤ 8. Likewise, the matrix representation of an interval graph can be used to characterize the distinct "meeting structure" of n segments.

***Index Terms*-** interval graph, meeting structure, matrix representation

I. INTRODUCTION

In a technological age, we have all become learners of mathematics. It is mathematical knowledge that has fuelled the fires of technological advances and it is those very technological advances that generate new and more complex mathematics. The task then of mathematics education is to help students and those who work with them to understand how to learn mathematics, how to solve problems, and how to acquire the automaticity with skills and procedures necessary for problem solving (Bayaga, 2007).

Mathematics is not only a conception of the relationship of numbers and the formations of objects in the space of the real world, but is also a technology that can be applied directly to solve issues in many aspects of the practice. Mathematics is a kind of sense and a mode of thought. We often solve problems by the standpoint of mathematics, by the viewpoint of mathematics and with the manners of mathematics, it is also a way of communication, for it provides a simple and precise information transmission. In conclusion, the matters, ideas, methods, and languages of mathematics had been widely applied in natural science, social science, and the operation of society as an important part of modern culture.

Mathematics provides an environment for logical, creative investigation of quantitative and relational situations. It consists of a large body of knowledge and many sub-disciplines, each of which provides an array of tools and techniques for exploration and analysis.

Combinatorics, as one of its discipline, is sometimes called the science of counting. It is concerned with the selection, arrangement and operation of elements within sets. It has applications in such diverse areas as managing computer and telecommunication networks, predicting poker hands and dividing tasks among workers. Among the many topics in combinatorics is meeting combinations and interval graphs (West, 1996).

An intersection graph is a graph that represents the pattern of intersections of a family of sets. Any graph may be represented as an intersection graph, but some special classes of graphs may be defined by the types of sets that are used to form an intersection representation of them and one of these classes are interval graphs.

An interval graph is the intersection graph of a set of intervals on a real line. It has one vertex for each interval in the set, and an edge between every pair of vertices corresponding to intervals that intersect. The path on each vertex are basically the "meeting structure" or "meeting combinations" of these graphs. Different patterns of intersection can occur so any pattern or arrangement establishes a set of "meeting" between two or more segments.

Interval graphs are chordal graphs and hence perfect graphs. By chordal graph, it means it does not contain an induced graph of length greater than three, while by perfect graph, it means the chromatic number of every induced subgraph is equal to the clique number of that subgraph. Their complements are comparability graphs, and the comparability relations are precisely the interval orders. Interval graphs are useful in modelling resource allocation problems in operations research as well as in linear scheduling problems having constraints on concurrent events. According to West (1996), some of the classical applications of interval graphs are analysis of DNA chains, phasing of traffic lights, archeological seriation and register allocation.

Objectives of the Study

This study aims to discuss the important concepts on meeting combinations and interval graphs. Specifically, it sought to

1. examine the properties of interval graphs
2. determine the number of different arrangements for n segments of distinct identities
3. determine the number of "meeting structures" or intersection for n segments
4. establish the matrix representation of an interval graph together with the associated characteristic polynomial.

Definition of Terms and Notations

A *graph* or undirected graph G is an ordered pair G = (V, E), such that V is a set whose elements are called vertices or nodes and E is a set of pairs (unordered) of distinct vertices called edges or lines.

The *valency* of a vertex is the number of edges connected to a vertex in a graph.

An m x n *matrix* is a rectangular array of mn real (or complex) numbers arranged in m horizontal rows and n vertical columns.

The *adjacency matrix* $M_G = (m_{ij})$ of G is defined as an n x n matrix whose elements are defined by

$$m_{ij} = \begin{cases} 1 \ if \ [v_1, v_2] \in E \\ 0 \ otherwise \end{cases}$$

The *characteristic polynomial* of a graph G, denoted by P(G: ʎ) is the characteristic polynomial of the adjacency matrix M_G of the graph G, that is

$$P(G: ʎ) = \det(ʎ \bullet 1 - M_G)$$

A *clique* is a set of vertices such that for every two vertices, there exists an edge connecting them. In other words, a clique is a complete graph.

The *clique number* of a graph G = (V, E), denoted by ω(G), is the size of the largest subset of V inducing a clique.

The *determinant* of an n – square matrix A, denoted as det(A) or |A| is

$$\sum 1^{\sigma} a_{1j_1} a_{2j_2} a_{3j_3} \dots a_{nj_n}$$

where the sum is taken all over permutations $\sigma = j_1 j_2 \dots j_n$ and

$$(1)^{\sigma} = \begin{cases} 1 \ if \ \sigma \ is \ even \\ -1 \ if \ \sigma \ is \ odd \end{cases}$$

An *interval graph* is an intersection graph of intervals on a real line where vertices are represented by intervals and there is an edge between two vertices if and only if their corresponding intervals intersect. Let $\{I_1, I_2, \dots, I_n\}$ be the set of intervals of real numbers, then the corresponding interval graph is G = (V, E) where $V = \{I_1, I_2, \dots, I_n\}$ and $E = \{\{I_\alpha, I_\beta\} / I_\alpha \cap I_\beta \neq \emptyset\}$.

Significance of the Study

The discussion and results of the study is of great importance to:

1. Analysis of DNA chains: Interval graphs were invented for the study of DNA. Benzer (1959) established the linearity of the chain for higher organisms. Each gene is encoded as an interval, except that the relevant interval may contain a dozen or more irrelevant junk pieces called "introns" among the relevant pieces called "exons". Under the hypothesis that mutations arise from alterations of connected segments, changes in traits of microorganisms can be studied to determine whether their determining amino-acid sets could intersect. This establishes a graph with traits as vertices and "common alteration" as edges. Under the hypothesis of linearity and contiguity, the graph is an interval graph, and this aids in locating genes along the DNA sequence (Berge, 1979).

2. Archeological seriation: Given pottery samples at an archeological dig, archaeologist seek a time-line of what styles were in use when. They assume that each style was used during a single time interval and that two styles appearing in the same grave were used concurrently. A graph is formed with the styles as vertices, making two styles adjacent if they appear together in a grave. If this is an interval graph, then its interval representation are the possible time-lines (Berge, 1979).

Furthermore, it is hoped that this study can stir up greater interest in mathematics. The elegance of the logical processes present the subject as a systematic process. In this way, teachers and mathematics enthusiasts may gain more insight in dealing with problems that may take a similar pattern as what are presented.

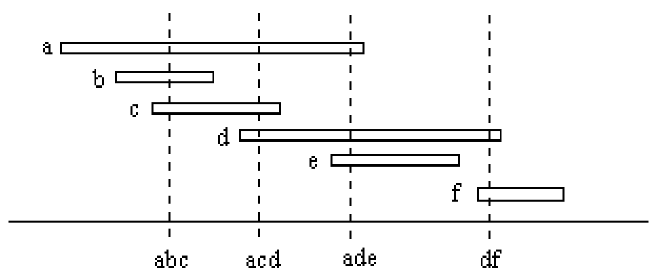
Research Design

This study will use descriptive research method, specifically on meeting combinations and interval graphs.

Results and Discussion

Let n continuous segments with freely defined starting and ending points be placed along the real line, and consider the intersection between these segments. Patterns of intersection can arise when n intervals of various lengths are placed with their starting points on the unit interval. For any given arrangement, there maybe some regions when no segment is present, some where just one segment is present, some where exactly two segments overlap, some where exactly three segments overlap, and so on. These arrangement establishes a set of "meetings" (intersection) between two or more segments. To illustrate, the figure shows the placements of six segments, and identifies the meetings between them.

Figure 1. The six interval segments on the real line.



The meeting for this arrangement are *abc*, *acd*, *ade*, and *df*. In general, the number of different arrangements of the endpoints for n segments (with distinct identities) is given in the following conjecture.

Theorem 1. Consider n distinct segments and let N_n denote the number of distinct arrangements of the n segments, then

$$N_n = n(2n-1)N_{n-1} = \frac{(2n)!}{2^n}$$

Proof:

For the first segment, there is obviously only one arrangement. This segment divides the axis into three regions, namely, the region to the left of the starting point, the region between the starting and ending points (i.e. the region covered by the segment), and the region to the right of the ending point. Therefore, the starting point of the second segment can be placed in any one of these regions, and the ending point can be placed in the same region or any other regions to the right of the region containing the starting point. Thus, there are six distinct ways of placing the second segment.

In general, as we prepare to add the n^{th} segment, there will already be $n - 1$ segments in place, and the $2(n - 1)$ terminal points of these segments divide the axis into $2n - 1$ regions. The starting point of the n^{th} segment can be placed in any of these regions, and then the ending point may be placed in the same or in a more rightward region. Thus, the number of possibilities for placing both these points is (2n – 1) + (2n – 2) + … + 2 + 1, which is equal to $n(2n - 1)$. It follows that if we let N_n denote the number of distinct arrangements of n segments, then we have

$$\boldsymbol{N_n = n(2n-1)N_{n-1}} = \frac{(2n)!}{2^n}$$

This applies only if the n segments have specified identities. If we relax this condition, allowing the labels of the segment to be interchangeable, then the number of distinct arrangement is much less. In general, we will have n starting points, so it only remains to place the n ending points. The only restriction is that each ending point must follow its starting point. Thus, the ending point for the first segment (i.e., the segment with the left-most starting point) can be places in any of n regions, delimited by the n starting points. We thus have the following corollary.

Corollary 2. Consider n indistinguishable segments and let N_n' denote the arrangement of n indistinguishable segments, then

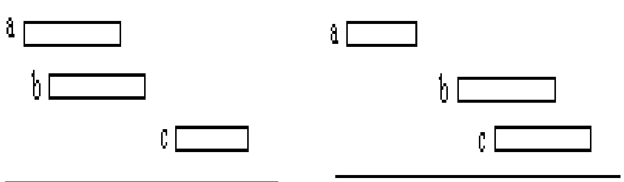
$$\boldsymbol{N_n' = n!}$$

Proof:

The ending point of the second segment can be placed in any of the n – 1 regions to the right of the second starting point, and so on. (The arrangement of the ending points in between consecutive starting points has no effect on the presence of intersections). Consequently, the number of arrangements for indistinguishable segments is simply **n!.**

However, this still distinguishes between the 1^{st} segment, the 2^{nd} segment, and so on. It is possible for two or more distinct arrangements to yield the same "meeting structures" up to permutation of the segments. For a trivial example, consider the two arrangements of three segments shown below.

Figure 2. Three interval segments on the real line.



(a) (b)

The segment labels merely indicate the order of the starting points, not implying any distinguishable identity for the segments.

In the arrangement shown in Figure 2(a) above, the meeting structure is *"ab"*, which signifies that the first and second segments overlap. In the arrangement in Figure 2(b), the meeting structure is *"bc"*, which signifies that the second and third segments overlap. If the orderings of the start times for the segments in the meeting structure are disregarded, then these two arrangements induce the same meeting structure. On this basis, what is the number of distinct meeting structures for n segments?

Let **M(n)** denote the number of distinct meeting structures for n segments. Obviously for n = 0 and n = 1 segments, no meetings are possible, so there is only one meeting structure in each of these cases, namely the null structure, denoted by {∅}. Thus we have M(0) = M(1) = 1.

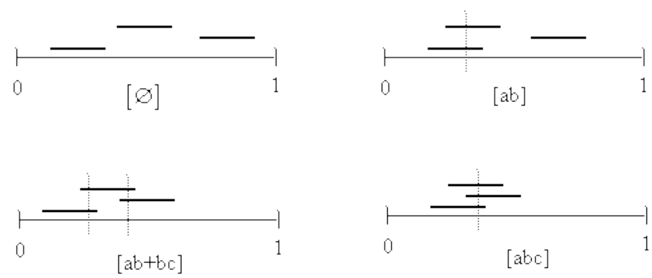
For n = 2 segments there are two possible meeting structures, corresponding to the fact that either these segments overlap or they do not. We will let {∅, ab} denote these two possibilities, and we have M(2) = 2.

For n = 3 segments, there are 3! = 6 distinct arrangements, since the number of permutation of n different things taken n at a time is n!. These arrangements can be expressed symbolically as

$$\emptyset, ab, bc, ab + bc, ab + ac, abc$$

However, as noted above, the arrangements ab and bc represent the same meeting structure. Likewise the arrangement $ab + bc$ is the same structure as $ab + ac$. In both cases, one segment meets with each of the other two, but the other two don't overlap with each other. Therefore, the number of distinct meeting structures for n = 3 segments is just four, consisting of the set {∅, ab, ab + bc, abc}, and so we have M(3) = 4. Examples of the four types of outcomes for n = 3 are shown on the figure.

Figure 3. The four meeting structures for three interval segments.



The expression inside the square brackets signify the structure of the set of intersection.

The meeting "abc" can also be written in the form *"ab + ac + bc"*, because these both signify a triple intersection between the segments a, b, and c. The conjoined characters signify a mutual meeting of these segments, and the "+" symbol represents logical "AND". Thus both of these operations are forms of "intersection". Since the basic sets are continuous segments on the one-dimensional x axis, the three pairwise meetings of three segments imply that the first ending point occurs after the last starting point, so there is a region on three-way intersection. However, the "meeting" relation between segments is not transitive, because ab + bc does not imply ac.

For n = 4 segments, we examine all n! = 24 distinct arrangements and finding out that there are only ten distinct meeting structures. These are

(1) ∅	(3) ab (c + d)
(3) ab	(5) a (bc + d)
(4) a (b + c)	(2) ab + bc + cd
(1) ab + cd	(2) abc
(2) a (b + c + d)	(1) abcd

(The number to the left of each structure is the number of occurrences of that structure in the n! arrangements.) Hence we have M(4) = 10.

To determine the value of M(5), we could examine each of the 5! = 120 arrangements by hand, and check for the unique structures, but this would be very laborious. To automate the process, we can step through each of the n! placements of the n ending points, and for each arrangement we can easily determine the pairwise meetings. We can then count how many pairs contain the first segment, how many for the second, and so on. We can then count how many segments appear in exactly one pair, how many in exactly two, and so on. This gives a set of numbers d_1, d_2, ... that characterize the meeting structure, independent of the orderings of the segments. If two structures are characterized by different values of d_j, then they are certainly distinct structures, so we can determine a lower bound on the value of M(n) by determining the number of unique characteristics $\{d_1, d_2, ...\}$.

For the case n = 5, this criterion turns out to be nearly sufficient to distinguish all the structures. There are 26 distinct sets of $\{d_j\}$ indices, so we know M(5) is at least 26. However, it is possible for distinct meeting structures to have the same set of $\{d_j\}$ values. This occurs for the structures *ab + ac + bc + de* and *ab + bc + cd + de*. In each case, two letters appear exactly once, and the other three letters appear twice each. However, they are obviously different structures, as seen by the fact that the first one reduces to *abc + de* whereas the second is irreducible. So, there are actually M(5) = 27 distinct meeting structures with n = 5. These are shown below.

Ø	a (b + c + d) + de	a (bc + de)
ab	a (bc + d)	a (b (c + d) + e)
ab + cd	abc + d (a + e)	a (b (c + d) + de)
a (b + c)	a (b + c + d) + c (d + e)	a (bcd + e)
a (b + c) + de	ab (c + d)	ab (c + d + e)
ab + ac + cd	ab (c + d) + de	ab (cd + e)
abc	abcd	abc (d + e)
abc + de	a (b + c + d + e)	ab + bc + cd + de
a (b + c + d)	a (bc + d + e)	abcde

To help distinguish structures that have the same $\{d_j\}$ sets, consider again the two structures *ab + ac + bc + de* and *ab + bc + cd + de*. If we replace each letter by the number of times in which it appears, these become 2•2 + 2•2 + 2•2 + 1•1 and 1•2 + 2•2 + 2•2 + 2•1 respectively, clearly distinct. Carrying out the multiplications and additions numerically, these give the values 13 and 12 respectively. These sums are an additional discriminator that can be used to help count the distinct meeting structures. On this basis, we can distinguish 91 meeting structures for n = 6. However, there are actually M(6) = 92 distinct meeting structures. The two that are indistinguishable based on the criteria described are *ab + bc + de + df + ef* and *ab + bc + cd + de + ef*. These have the same set of valences, and the same pairing of valences, but they are nevertheless distinct structures, as shown by the fact that the first can be reduced to *ab + bc + def*, whereas the second is irreducible.

We can define another discriminator by replacing each letter not with the valence of that letter, but with the sum of the valences of all the letters with which it is paired. For example, in the first structure e is paired with d and f, and these two letters appear twice each, for a total of 4, so we replace the letter e with 4. Doing the same for all the other letters in these structures, and carrying out the multiplications and additions numerically, we get

2•2 + 2•2 + 4•4 + 4•4 + 4•4 = 56

2•3 + 3•4 + 4•4 + 4•3 + 3•2 = 52

By this criterion, it enables us to distinguish between these two structures. On the same basis we can determine that M(7) = 369, meaning that there are 369 distinct meeting structures for n = 7 segments. However, for n = 8 segments the criteria we have described so far are not sufficient to distinguish between all the different structures. We find that there are at least 1800 such structures by these criteria, but in fact there are 1807. It is interesting to try to identify the most efficient discriminators between these structures. Obviously if we are given two sets of pairing expressions we can simply try all possible permutations of the letters and terms, to see if they can be brought into agreement, but this in not feasible for large values of n.

Table 1. Sample values of the number of distinct meeting structures for n segments:

n	M(n)
0	1
1	1
2	2
3	4
4	10
5	27
6	92
7	369
8	1870

Matrix Representation of the Meeting Structure of n Segments:

There are several different ways to represent a graph in a computer. Although graphs are usually shown diagrammatically, this is only possible when the number of vertices and edges is reasonable small. Graphs can also be represented in the form of matrices. The major advantages of matrix representation is that, the calculation of paths and cycles can easily be performed using well known operations on matrices. Any graph of n vertices with an n x n symmetric matrix is called the graph's adjacency matrix. Each column and each row of the adjacency matrix represents a vertex. A 1 is placed in a position if the two vertices that the position represents are connected, a 0 is placed in that position if the two vertices are not connected. In general, the number of 1's in the k^{th} row/column corresponds to the number of edges adjacent to v_k.

One way of characterizing the unique meeting structures for n segments is in terms of the determinant of a certain n x n matrix.

Let a_k with k = 1, 2,.., n designate the n segments, and define an n x n matrix A with the non-zero components $A_{j,k} = 1$ for each pair of indices j, k such that $a_j a_k$(or $a_k a_j$) is a connected pair. In addition, we set the diagonal values of A to the parameter x, so we have $A_{j,j} = x$ for j = 1 to n.

Note that a meeting structure is specified as a union of m pair-wise connections. If no two segments meet, then m = 0. If there is a connection between every pair of segments, then m = n (n-1)/2.

Given the matrix A for a particular structure, the determinant of A is a polynomial in the parameter **x**. Since the determinant of a matrix is invariant under permutations of the row/columns, it follows that this polynomial characterizes a full equivalence class of meeting structures. Thus we can use these polynomials to discriminate between distinct structures. To illustrate, consider the following two meeting structures for n = 7 segments:

(1) $a_2 a_3 + a_3 a_4 + a_3 a_5 + a_4 a_5 + a_4 a_6 + a_4 a_7 + a_5 a_6 + a_5 a_7$
(2) $a_2 a_3 + a_2 a_4 + a_2 a_5 + a_2 a_6 + a_3 a_4 + a_3 a_5 + a_5 a_6 + a_5 a_7$

These structures have the same valence sets (i.e., they each have two characters that appear in four pairs, one that appears in three pairs, two that appear in two pairs, and one that appears in one pair), so in order to distinguish between them using the previous method we need to go to higher-order valences. However, these structures correspond to the matrices

Figure 4.

$$\begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & x & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & x & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & x \end{bmatrix} \qquad \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & x & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & x & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & x & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & x \end{bmatrix}$$

(a) (b)

and the determinants of the above matrices are

$$x^2 (x - 1) (x^4 + x^3 - 7x^2 - x + 4) \text{ for Fig.4(a)}$$
$$x (x^6 - 8x^4 + 6x^3 + 7x^2 - 4x - 1) \text{ for Fig.4(b)}$$

This shows how we can immediately distinguish between these two structures. (For automated numerical purposes, we can simply evaluate these polynomials for a few arbitrary values of x to discriminate between them).

The matrices and characteristic polynomials for the four distinct meeting structures for n = 3 segments are shown below.

Figure 5.

$$\det\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} = x^3 \qquad \det\begin{bmatrix} x & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & x \end{bmatrix} = x\,(x^2 - 1)$$

(a) (b)

$$\det\begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix} = x\,(x^2 - 2) \qquad \det\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{bmatrix} = (x - 1)^2\,(x + 2)$$

(c) (d)

Likewise we can determine the characteristic polynomials for each of the 10 interval graphs on n = 4 nodes, shown below

∅	x^4
ab	$x^2 (x – 1) (x + 1)$
ab + bc	$x^2 (x^2 – 2)$
ab + cd	$(x – 1)^2 (x + 1)^2$
abc	$x (x + 2) (x – 1)^2$
ab + bc + cd	$(x^2 + x – 1) (x^2 – x – 1)$
abc + cd	$(x – 1) (x^3 + x^2 – 3x – 1)$
abc + abd	$x (x – 1) (x^2 + x – 4)$
ab + ac + ad	$x^2 (x^2 – 3)$
abcd	$(x + 3) (x – 1)^3$

As noted previously, the only other plane graph on four nodes is the four-loop

ab + bc + cd + da $x^2 (x – 2) (x + 2)$

which is not an interval graph.

Summary and Conclusions

This research was conducted to develop the key concepts on meeting combinations and interval graphs. It specifically aimed to (1) examine the properties of interval graphs; (2) determine the number of different arrangements for n segments of distinct identities; (3) determine the number of "meeting structures" or intersection for n segments and (4) establish the matrix representation and characteristic polynomial of a graph. Based from the results of this research, the vertices of an interval graph G can be ordered such that v_i is connected to v_k implies v_j is connected to v_k whenever i < j < k; the number of different arrangements of the endpoints for n segments is n!, but since some of these arrangements yields the same intersection, then the number of possible distinct "meeting structure" for these segments will be reduced to M(n); the matrix representation of an interval graph can be used to characterize the unique "meeting structures" of these segments.

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