Strong Split Domination Polynomial of Cycles

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Abstract

Let $G = (V(G), E(G))$ be a simple graph. A dominating set $D \subseteq V(G)$ is a strong split dominating set if the induced subgraph $< V - D >$ is totally disconnected with at least two vertices. Let $\mathcal{D}_{ss}(G, i)$ be the family of strong split dominating sets of G of cardinality i and $|\mathcal{D}_{ss}(G, i)| = d_{ss}(G, i)$. We define the strong split domination polynomial of a graph G of order n as the polynomial$D_{ss}(G, x) = \sum_{i=\gamma_{ss}(G)}^{n-2} d_{ss}(G, i)x^i$. In this paper, we determine the strong split domination polynomial of cycles and obtain some of its properties.

Keywords and Phrases: Strong split dominating set, Strong split domination polynomial.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a γ-set. For a detailed treatment of this parameter the reader is referred to [2]. In [1], S Alikhani and Y H Peng has found the recursive relation for the domination polynomial of cycles. Now in the same way we find the recursive relation for the strong split domination polynomial of cycles.

A dominating set $D \subseteq V(G)$ is a strong split dominating set if the induced subgraph $< V - D >$ is totally disconnected with at least two vertices. The strong split domination number is the minimum size of a strong split dominating set of G and is denoted by $\gamma_{ss}(G)$. Strong split domination in graph was introduced by Kulli and Janakiraman in [3]. For more details on strong split domination we refer [4]. It is immediate that for any cycle C_n, $\gamma_{ss}(C_n) = \left\lceil \frac{n}{2} \right\rceil$ [4].

Definition 1.1 [5] Let $\mathcal{D}_{ss}(G,i)$ be the collection of strong split dominating sets of G of cardinality i and $|\mathcal{D}_{ss}(G,i)| = d_{ss}(G,i)$. The strong split domination polynomial of G is defined as $D_{ss}(G,x) = \sum_{i=\gamma_{ss}(G)}^{n-2} d_{ss}(G,i)x^i$.

2. Construction of Strong Split Dominating Sets of Cycles

A Cycle is a graph whose vertices can be listed in the order $\{u_1, u_2, \dots, u_n\}$ such that the edges are $\{(u_1,u_2),(u_2,u_3),,\dots,(u_{n-1},u_n),(u_n,u_1)\}$. Let $\mathcal{D}_{ss}(C_n,i)$ be the collection of strong split dominating sets of C_n with cardinality i.

Observation 2.1 *For any cycle C_n,*

1. $\mathcal{D}_{ss}(C_n,i) = \emptyset$ *if and only if* $i > n-2$ *or* $i < \left\lceil \frac{n}{2} \right\rceil$
2. $\mathcal{D}_{ss}(C_n,i) \neq \emptyset$ *if and only if* $\left\lceil \frac{n}{2} \right\rceil < i < n-2$
3. *To find a strong split domination polynomial of C_n with cardinality i, it is enough to consider $\mathcal{D}_{ss}(C_{n-1},i-1)$ and $\mathcal{D}_{ss}(C_{n-2},i-1)$. Thus we have to consider four combinations of whether these two collections are empty or not.*

Lemma 2.1 If $\mathcal{D}_{ss}(C_{n-1},i-1) = \emptyset$ and $\mathcal{D}_{ss}(C_{n-2},i-1) = \emptyset$, then $\mathcal{D}_{ss}(C_n,i) = \emptyset$.

Lemma 2.2 *Suppose $\mathcal{D}_{ss}(C_n,i) \neq \emptyset$. Then*

1. $\mathcal{D}_{ss}(C_{n-1},i-1) = \emptyset$ *and* $\mathcal{D}_{ss}(C_{n-2},i-1) \neq \emptyset$ *if and only if* $n = 2k$ *and* $i = k$ *for some* $k \in \mathbb{N}$.
2. $\mathcal{D}_{ss}(C_{n-1},i-1) \neq \emptyset$ *and* $\mathcal{D}_{ss}(C_{n-2},i-1) = \emptyset$ *if and only if* $i = n-2$.
3. $\mathcal{D}_{ss}(C_{n-1},i-1) \neq \emptyset$ *and* $\mathcal{D}_{ss}(C_{n-2},i-1) \neq \emptyset$ *if and only if* $\left\lceil \frac{n-1}{2} \right\rceil + 1 \leq i \leq n-2$.

Proof. 1. Assume that $\mathcal{D}_{ss}(C_{n-1},i-1) = \emptyset$ and $\mathcal{D}_{ss}(C_{n-2},i-1) \neq \emptyset$. Since $\mathcal{D}_{ss}(C_{n-1},i-1) = \emptyset$ by Observation 2.1, $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$. If $i-1 > n-3$, then $i > n-2$ and by Observation 2.1, $\mathcal{D}_{ss}(C_n,i) = \emptyset$ a contradiction. So $i < \left\lceil \frac{n-1}{2} \right\rceil + 1$ and since $\mathcal{D}_{ss}(C_n,i) \neq \emptyset$ together with $\left\lceil \frac{n}{2} \right\rceil \leq i < \left\lceil \frac{n-1}{2} \right\rceil + 1$, we have $n = 2k$ and $i = k$ for some $k \in \mathbb{N}$.
Conversely, if $n = 2k$ and $i = k$ for some $k \in \mathbb{N}$, then by Observation 2.1, $\mathcal{D}_{ss}(C_{n-1},i-1) = \emptyset$ and $\mathcal{D}_{ss}(C_{n-2},i-1) \neq \emptyset$.

2. Assume that $\mathcal{D}_{ss}(C_{n-1},i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(C_{n-2},i-1) = \emptyset$. Since $\mathcal{D}_{ss}(C_{n-2},i-1) = \emptyset$, by Observation 2.1, $i-1 > n-4$ or $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$. If $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$, then $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$ and hence $\mathcal{D}_{ss}(C_{n-1},i-1) = \emptyset$, a contradiction. So $i > n-3$ and also since $\mathcal{D}_{ss}(C_{n-1},i-1) \neq \emptyset$, $i-1 \leq n-3$. Therefore, $i = n-2$.

Conversely, if $i = n-2$, then by Observation 2.1, $\mathcal{D}_{ss}(C_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(C_{n-2}, i-1) = \emptyset$.

3. Let us assume that $\mathcal{D}_{ss}(C_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(C_{n-2}, i-1) \neq \emptyset$. Then by Observation 2.1, $\left\lceil\frac{n-1}{2}\right\rceil \leq i-1 \leq n-3$ and $\left\lceil\frac{n-2}{2}\right\rceil \leq i-1 \leq n-4$. So, $\left\lceil\frac{n-1}{2}\right\rceil \leq i-1 \leq n-4$ and hence $\left\lceil\frac{n-1}{2}\right\rceil + 1 \leq i \leq n-2$.

Conversely, if $\left\lceil\frac{n-1}{2}\right\rceil + 1 \leq i \leq n-2$, then by Observation 2.1, $\mathcal{D}_{ss}(C_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(C_{n-2}, i-1) \neq \emptyset$.

Theorem 2.1 *Let* $n \geq 6$ *and* $i \geq \left\lceil\frac{n}{2}\right\rceil$

1. *If* $\mathcal{D}_{ss}(C_{n-1}, i-1) = \emptyset$ *and* $\mathcal{D}_{ss}(C_{n-2}, i-1) \neq \emptyset$, *then* $\mathcal{D}_{ss}(C_n, i) = \{\{1,3,5,\ldots,-1\},\{2,4,6,\ldots,n\}\}$.
2. *If* $\mathcal{D}_{ss}(C_{n-1}, i-1) \neq \emptyset$ *and* $\mathcal{D}_{ss}(C_{n-2}, i-1) = \emptyset$, *then* $\mathcal{D}_{ss}(C_n, i) = \mathcal{S} \cup \{\{1,2,3,\ldots,n\} - \{x,y\}/x \text{ and } y \text{ are not adjacent}\}$ *where* $\mathcal{S} = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(C_{n-1}, n-3)\}$.
3. *If* $\mathcal{D}_{ss}(C_{n-1}, i-1) \neq \emptyset$ *and* $\mathcal{D}_{ss}(C_{n-2}, i-1) \neq \emptyset$, *then* $\mathcal{D}_{ss}(C_n, i) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, *where* $\mathcal{S}_1 = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(C_{n-1}, i-1)\}$ *and* $\mathcal{S}_2 = \{X_2 \cup \{n-1\}/X_2 \in \mathcal{D}_{ss}(C_{n-2}, i-1)\}$ *and* $\mathcal{S}_3 = \{X_3 \cup \{n\}/X_3 \in \mathcal{D}_{ss}(C_{n-2}, i-1) - \mathcal{D}_{ss}(C_{n-1}, i-1)\}$

Proof. 1. Let $\mathcal{D}_{ss}(C_{n-1}, i-1) = \emptyset$ and $\mathcal{D}_{ss}(C_{n-2}, i-1) \neq \emptyset$. By Lemma 2.2(1) $n = 2k$ and $i = k$ for some $k \in \mathbb{N}$. Then $\mathcal{D}_{ss}(C_n, i) = \mathcal{D}_{ss}(C_{2k}, k) = \{\{1,3,5,\ldots,-1\},\{2,4,6,\ldots,n\}\}$.

2. Let $\mathcal{D}_{ss}(C_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(C_{n-2}, i-1) = \emptyset$. By Lemma 2.2(2), $i = n-2$. Therefore $\{\{1,2,3,\ldots,n\} - \{x,y\}/x \text{ and } y \text{ are not adjacent}\}$, be the collection of strong split dominating sets of C_n of cardinality $n-2$ and $\mathcal{S} = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(C_{n-1}, n-3)\}$.

3. Let $\mathcal{D}_{ss}(C_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(C_{n-2}, i-1) \neq \emptyset$ ad assume that $X_1 \in \mathcal{D}_{ss}(C_{n-1}, i-1)$. Then $X_1 \cup \{n\} \in \mathcal{D}_{ss}(C_n, i)$. Take $\mathcal{S}_1 = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(C_{n-1}, i-1)\}$. Then $\mathcal{S}_1 \subseteq \mathcal{D}_{ss}(C_n, i)$.

Let us assume that $X_2 \in \mathcal{D}_{ss}(C_{n-2}, i-1)$. Then $X_2 \cup \{n-1\} \in \mathcal{D}_{ss}(C_n, i)$. Take $\mathcal{S}_2 = \{X_2 \cup \{n-1\}/X_2 \in \mathcal{D}_{ss}(C_{n-2}, i-1)\}$. Then $\mathcal{S}_2 \subseteq \mathcal{D}_{ss}(C_n, i)$. Thus $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{D}_{ss}(C_n, i)$.

Now let us assume that $X_3 \in \mathcal{D}_{ss}(C_{n-2}, i-1) - \mathcal{D}_{ss}(C_{n-1}, i-1)$. Then $X_3 \cup \{n\} \in \mathcal{D}_{ss}(C_n, i)$. Take $\mathcal{S}_3 = \{X_3 \cup \{n\}/X_3 \in \mathcal{D}_{ss}(C_{n-2}, i-1) - \mathcal{D}_{ss}(C_{n-1}, i-1)\}$. Then $\mathcal{S}_3 \subseteq \mathcal{D}_{ss}(C_n, i)$. Thus $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \subseteq \mathcal{D}_{ss}(C_n, i)$.

Now suppose that $Z \in \mathcal{D}_{ss}(C_n, i)$. Then $n \in Z$ or $n \notin Z$.

If $n \in Z$, then there exist $X_1 \in \mathcal{D}_{ss}(C_{n-1}, i-1)$ such that $Z = X_1 \cup \{n\}$ and $X_3 \in \mathcal{D}_{ss}(C_{n-2}, i-1) - \mathcal{D}_{ss}(C_{n-1}, i-1)$such that $Z = X_3 \cup \{n\}$. Hence $Z \in \mathcal{S}_1 \cup \mathcal{S}_3$. If $n \notin$

Z, then $n-1 \in Z$ otherwise $Z \notin \mathcal{D}_{ss}(C_n, i)$. If $n-1 \in Z$, then there exist $X_2 \in \mathcal{D}_{ss}(C_{n-2}, i-1)$ such that $X_2 \cup \{n-1\} \in Z$. Thus $Z \in \mathcal{S}_2$.
Therefore $\mathcal{D}_{ss}(C_n, i) \subseteq \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.
Hence the Proof.

3. Strong Split Domination Polynomial of Cycles

In this section we determine the strong split domination Polynomial of Cycles and some of its properties.

Definition 3.1 Let $\mathcal{D}_{ss}(C_n, i)$ be the collection of strong split dominating sets of C_n of cardinality i and $|\mathcal{D}_{ss}(C_n, i)| = d_{ss}(C_n, i)$. Then the strong split domination polynomial of cycle is defined as $D_{ss}(C_n, x) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} d_{ss}(C_n, i)x^i$.

Theorem 3.1 *If $\mathcal{D}_{ss}(C_n, i)$ is the collection of strong split dominating set of cardinality i of C_n ,then $|\mathcal{D}_{ss}(C_n, i)| = |\mathcal{D}_{ss}(C_{n-1}, i)| + |\mathcal{D}_{ss}(C_{n-2}, i)| + |\mathcal{D}_1| + |\mathcal{D}_2|$ where $\mathcal{D}_1 = \{\{1,2,3, \dots, n\} - \{x, y\}/x \text{ and } y \text{ are not adjacent}\}$ and $\mathcal{D}_2 = \{\{1,2,3, \dots, n\} - \{1, n-1, u\}/u \in \mathcal{D}_{ss}(C_{n-2}, i-1) - \mathcal{D}_{ss}(C_{n-1}, i-1)\}$.*

Proof. By using Theorem 2.1, the result follows.

Theorem 3.2 *For every Cycle $C_n (n \geq 6)$,*
$D_{ss}(C_n, x) = x\left(D_{ss}(C_{n-1}, x) + D_{ss}(C_{n-2}, x)\right) + (n-5)x^{n-3} + (n-2)x^{n-2}$ *with* $D_{ss}(C_4, x) = 2x^2$ *and* $D_{ss}(C_5, x) = 5x^3$.

Proof. By using Theorem 3.1 and the definition of strong split domination polynomial we get the result.

Theorem 3.3 *Let $D_{ss}(C_n, x)$ be the strong split domination polynomial of cycle C_n. Then the following properties hold.*

1. *For any positive integer n, $d_{ss}(C_n, n-1) = 0$ and $d_{ss}(C_n, n) = 0$.*
2. $d_{ss}(C_n, i) = d_{ss}(C_{n-1}, i-1) + d_{ss}(C_{n-2}, i-1)$, *for any positive integer* $\lceil \frac{n}{2} \rceil \leq i \leq n-4$.
3. $d_{ss}(C_{2n}, n) = 2$, *for every positive integer $n \geq 2$.*
4. $d_{ss}(C_n, n-2) = \frac{n(n-3)}{2}$, *for every positive integer $n \geq 3$.*

Proof. 1. The result follows from Definition 3.1.

2. It follows from Theorem 3.2.

3. By Theorem 2.1(1), $\{\{1,3,5, \dots, 2n-1\}\{2,4,6, \dots, 2n\}\}$ is the only strong split dominating set of size n. Hence $d_{ss}(C_{2n}, n) = 2$.

4. There are $\binom{n}{n-2}$ sets of cardinality $n-2$. In any cycle C_n, exactly n pair of vertices are adjacent. So, the number of strong split dominating sets of cardinality $n-2$ will be $d_{ss}(C_n, n-2) = \binom{n}{n-2} - (n) = \frac{n(n-3)}{2}$.

4. Conclusion

In this paper we have found the Strong Split Domination polynomial for Cycles. In future we plan to investigate the polynomial for several graph products.

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