# A New Strategy for the Re-Initialization of the Level-Set Field: The Lagrange Method of a Multiplier

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**ABSTRACT**

In this paper, a novel strategy has been introduced for the re-initialization of the Level-Set field by employing the Lagrange method of a multiplier approach. The introduced scheme is quite efficient, capable, and based on the Eulerian-Lagrangian method of a multiplier. The idea of a geometric-based re-initialization scheme is merged using the concept of the Finite Element Method (FEM) and then implemented to higher degree polynomials. Numerical test examples will be demonstrated that serve the effectiveness and efficacy of the introduced scheme.

**Keywords:** Re-initialization, The Level-Set method, discontinuous Galerkin method, The Finite Element Method, unstructured grid, multiphase flows, and the Lagrange method of a multiplier.

## 1) INTRODUCTION

The flow that consists of more than one phase is called Multiphase Flow. Multiphase flow has many applications and, it is being used in the different types of industrial applications that are varying in forms from power generations to food, beverages, and the medical sciences etc. There are two popular and well-known techniques for the representation of the multiphase flow.

i. Eulerian-Eulerian (this technique is used usually in the study of the dense dispersed system).
ii. Eulerian-Lagrangian (it is the more suitable technique for the particle transport examples).

Both methods mentioned above permit the interaction of stages in terms of momentum, mass exchange heat, and turbulence. In physics, a numerical representation should be able to find out that includes drag calculations, free-surface flows, particle transport etc.

In a model of two-phase flow, the two channels in addition to the distinct characteristics and distinct phases exist in a single domain. The flow of this type shows the main part in several industrial applications that comprises chemical reactors, medical sciences, petroleum industries etc. The two-phase flow dynamics may be sophisticated due to the moving interface the two channels change their material characteristics in time and space. In addition, by the confined flow, the interface is not advected in many cases. However, the shortcoming intermolecular forces (tension of the surface) on the interface amid the two channels be the cause of the confined flow of acceleration. The two-phase can be classified into four types; the types are (i)

Liquid-Gas flow, (ii) Liquid-Solid flow, (iii) Solid-Gas flow and, (iv) Liquid-Liquid flow. In all of the four types, the characteristics of the material are different from each other and the motion of the interface shows the main part due to the well-defined dissimilar material characteristics of two phases, for instance, density and viscosity.

The essential parameters devised for the system of the two-phase flow include Heat transfer coefficient, Flux limitations, Pressure drop, Mean phase content and Mass transfer coefficient. The industrial and engineering applications of the two-phase flow are steam generators and condensers, steam turbines (power plants), coal-fired furnaces, liquid sprays, refrigeration, pumping of flashing liquids, oil industry where two-phase arise that carries natural gas and oil, paper production, free-surface flows where well-defined interface exists energy conversion, food production and raining bed driers

For the modelling of the systems of the two-phase flow, an extensive variety of the models have been formulated including computational fluid dynamics (CFD) models, separated flow models, drift-flux models, homogenous models and multi-fluid models. The well-known and famous methods for the modelling of the two-phase flow are the Level-Set (LS) method [17], Volume of Fluid (VOF) method [19], the modified Level-Set (MLS) method [12], Lattice Boltzmann method [8], The Marker Particle method [18], and Smooth Particle Hydrodynamics method [7].

In Section 2 of this paper, the Level-Set method is discussed, In Section 3; the formulation of the Discontinuous Galerkin (DG) method has described. In Section 4, the new re-initialization scheme is described along with its example. Test problems have been presented in Section 5 and Section 6 (Last Section) the conclusion of the proposed re-initialization scheme is presented.

## 2) THE LEVEL-SET METHOD

In 1987 the two American mathematicians *Stanley Osher* and *James Seithan* presented the Level-Set Method. The Level-Set Method (LSM) acts as a very robust numerical method and this method is devised to detect the moving interfaces. In the Level-Set method, the detection of the moving interface is easy to show but mass does not conserve (mass conservation issues arise) so, this method is much better to track the moving interface. The solution of the geometric partial differential equation (PDE) can usually be the interfaces; the main concept of the Level-Set Method (LSM) acts to depict the interface completely by way of the zero Level-Set illustrated in the high dimensional Euclidean space.

In 2D (two-dimensional case), it is assumed that the moving curve (interface) is defined by $\Gamma(t)$ (or in a 3D surface) bounded over the region $\Omega \in \Re^2$ (it is not required that bounded region is closed). The movement of the moving curve (interface) is driven by the velocity field $\underline{V} = (v_1, v_2)$ that relies on its location, time, and geometry.

The basic idea explained above is to introduce the Level-Set method as a Level-Set function $\phi(x, y, t)$ in the one-dimensional case and higher, it has the property that it is negative, in one region and positive, on the other one and its contour is at zero is $\phi(x, y, t)$. The present location of the moving curve, (interface) $\Gamma(t) = \{(x, y) \mid \phi(x, y, t)\}$ is always depicted by the contour is at zero.

Mathematically, the equation of the Level-Set function is defined as

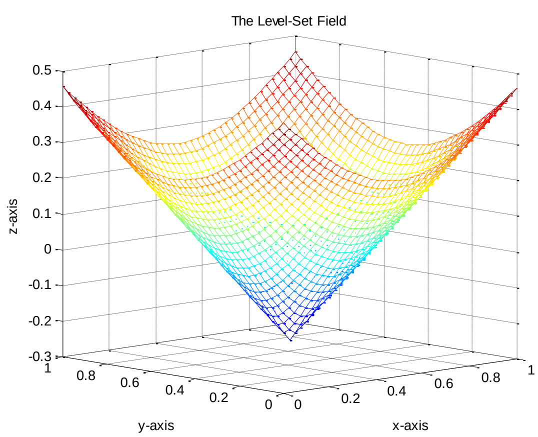
$$\phi_t + \underline{V} \cdot \nabla \phi = 0 \quad (1)$$

Where $\phi_t$ represents the changing of interface with respect to time, $\underline{V}$ is divergence free velocity field and $\nabla \phi$ is the gradient of a sign distance function of the interface.

It is ascertained that the Level-set function preserves the sign distance property i.e. $|\nabla \phi| = 1$and $\phi$is a sign distance function of the moving curve (interface). In the two-dimensional case, the magnitude of the sign distance function of the Level-Set method can be calculated as

$$\sqrt{\phi_x^{\ 2} + \phi_y^{\ 2}} \quad (2)$$

Where $\phi$ represents the Level-Set values and $x$, $y$ shows its directions.



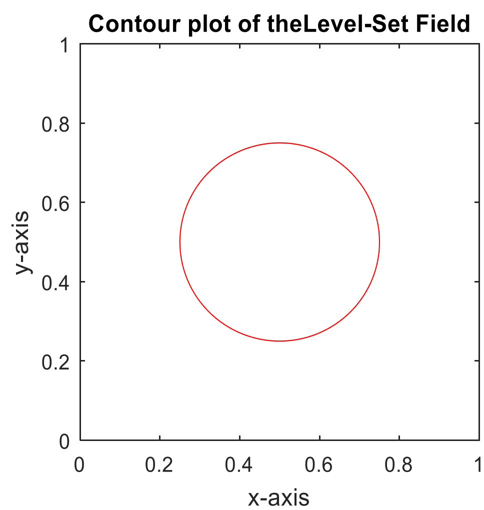




**Figure. 1**– Instance of the Level-Set method

## 3) FORMULATION OF THE DISCONTINUOUS GALERKIN METHOD

Discontinuous Galerkin was initially examined and presented at the beginning of the 1970s as a numerical technique to solve the partial differential equation (PDE). In this method, the characteristics of the Finite Volume and Finite Element structures are combined and successfully it has been implemented in parabolic, elliptic, hyperbolic, and in mixed problems that appear in an extensive variety of the applications

### 3.1)Discretization of the discontinuous Galerkin Method of the Level-Set equation

In the discretization scheme of the discontinuous Galerkin (DG) method, the merits of the Finite Element and the Finite Volume approaches are combined. Inside every element, its solution is extended on the basis of the polynomial, and the flux at each interface is specified individually between the two elements but not the solution, In fact, the solution is not defined at each interface. It means that the interface is completely continuous piecewise only i.e. (element to element-wise) using the normal vector and curvature of the interface due to the combination of the flow equations to the model of the interface.

### 3.2) Spatial Discretization

In a spatial discretization, split up the computational domain $\Omega$ in a set of $N_T$ rectilinear triangle volume control $\Omega_k$, this procedure is known as *tessellation*. Tessellation may automatically be done if these types of volume controls are selected for the domain as depicted in **Figure 2**

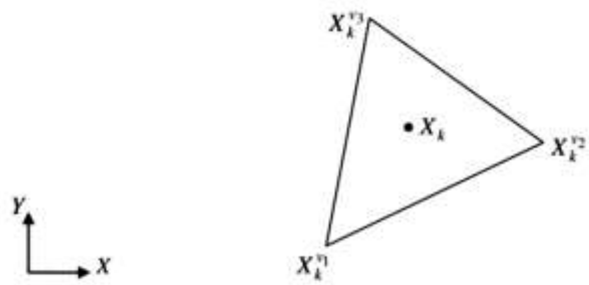



**Figure 2:** Triangular Volume Control $(\Omega_k)$

Like a basis function, the n[th] degree of the Legendre polynomial $L_i(x)$ is used to extend the solution inside every element is,

$$\phi_k^h(x,t) = \sum_{i=1}^{n_p} \phi_i(t)\, L_i(x), \quad x \in \Omega_k \tag{3}$$

Where,

$n_p$ is the total number of the nodal points inside every single element.

The solenoid velocity $u_\alpha$ is given as

$$\frac{\partial \Phi(x)}{\partial t} + (u_\alpha \Phi(x))_\alpha = 0, \quad x \in \Omega \tag{4}$$

Where $u_\alpha$ represents the interface velocity. The eq. (4) shows that the Level-Set function is preserved. The best approximation of the eq. (4) may be determined by levying the substitution of the eq. (3) the residuals are orthogonal to the space polynomial, and it is spanned using the increment of the solution. It is defined as

$$\int_{\Omega_k} \left( \frac{\partial \phi_k^h(x,t)}{\partial t} + \nabla \cdot (u(x,t)\, \phi_k^h(x,t)) \right) L_i(x)\, d\Omega = 0, \quad 1 \le i \le n_p \tag{5}$$

Implement the technique of integration by parts, the deficient form of the eq. (4) is attained through the Gauss divergence theorem as

$$\int_{\Omega_k} \left( \frac{\partial \phi_k^h(x,t)}{\partial t} L_i(x) - \nabla L_i(x) \cdot (u(x,t)\, \phi_k^h(x,t)) \right) d\Omega = -\oint_{\partial\Omega_k} n \cdot \left( u\, \phi_k^h \right)^* L_i(x)\ d\Omega \tag{6}$$

Where

$\hat{n}$ shows the normal of pointing outward and $\phi_k^h$ shows the numerical flux

Pointing outward normal and the numerical flux both are used as boundary conditions on each element. By using the integration by parts technique to make the deficient form into an efficient form is,

$$\int_{\Omega_k} \left( \frac{\partial \phi_k^h(x,t)}{\partial t} + \nabla \cdot (u(x,t)\, \phi_k^h(x,t)) \right) L_i(x)\, d\Omega = \oint_{\partial\Omega_k} n \cdot \left( u\, \phi_k^h - (u\phi_k^h) \right)^* L_i(x)\ d\Omega \qquad (7)$$

In the definition of the discontinuous Galerkin approach, the main parameter is to select the numerical flux and there is a small number of chances for scalar transport equation (4) to analyze. The Lax-Friedrichsand the central approximation are determined and presented in [1]. Numerical trials demonstrate that once a central flux approximation is applied, the motion will happen. The approximation of the numerical flux is specified in Eq. (7) and presently it may be appeared in the system of linear equations and levied the orthogonality on it for the solution of the $n_p$ nodal points it introduces the system of $n_p$ linear equations as,

$$M^k \frac{\partial \Phi}{\partial t} + S^k \cdot (u\Phi) = F^k (n \cdot (u\Phi - (u\Phi)^*)) \qquad (8)$$

Where,

$\Phi = \left( \phi_1, \phi_2, \text{K K}, \phi_{n_p} \right)$, $u = (u_1(t), u_2(t), \text{K K}, u_{n_p}(t))$ are the nodal vector values of $\phi_k^h(x,t)$ and at time $t\, u(x,t)$ inside element $\Omega_k$ respectively. $M^k$ shows the matrix of *mass*, $S^k$ shows the matrix of *Stiffness* and $F^k$ shows the element operator that operates boundary and it is determined as

$$M_{ij}^k = \int_{\Omega_k} L_j(x) L_i(x)\, d\Omega \qquad S_{ij}^k = \int_{\Omega_k} \nabla L_j(x) L_i(x)\, d\Omega \qquad F_{ij}^k = \oint_{\partial\Omega_k} L_j(x) L_i(x)\, \partial\Omega$$

For every element, the eq. (8) is solved, but without studying the solution it may not be solved inside the neighboring elements. The dimensions of all the systems are $n_p \times n_p$ and it may be formed in the one linear system of an ordinary differential equation. Mathematically it is expressed as

$$A \frac{\partial \overline{\Phi}(t)}{\partial t} + B\overline{\Phi}(t) = g(t) \qquad (9)$$

Where, $\overline{\Phi}(t) = \left( \phi_1(t), \phi_2(t), \text{K K}, \phi_{n_p * N_T}(t) \right)$ and $g(t)$ shows the offering of the not homogenous boundary conditions.

## 4) NEW RE-INITIALIZATION SCHEME

In this section, a new re-initialization scheme is presented, which is investigated in this research. In the Level-Set method issues of re-initialization arise to cope with that problem many methods have been presented in past such as the partial differential equation (PDE) based re-initialization method [3], the mass-preserving geometry-based re-initialization method [4] etc.

In this paper, the geometric-based re-initialization scheme is presented and the concept of finite element analysis is implemented in it. The concept is that a second-degree polynomial has fitted over the rectangular element of the domain and the shortest distances are calculated by using the Lagrange method of multiplier approach from the polynomial to all the nodes of each element. It is noted that the sign of the Level-Set values would not be changed above and below the interface i.e. $sign(\phi)$.where $\phi$ are the Level-Set values and sign represented by (negative and positive) signs from the interface. After calculating the shortest distance from polynomial to all the nodes of each element, so, the new updated value of the Level-Set field is expressed as

$$\phi' = sign(\phi) \cdot d \tag{10}$$

Where $\phi'$ are the new updated Level-Set values, $\phi$ are the actual Level-Set values, and the shortest distance from the interface is indicated by *d*. In the finite element analysis, the equation of the polynomial over the rectangular element is written as

$$\phi(x, y) = N_1\,\phi_1 + N_2\,\phi_2 + N_3\,\phi_3 + N_4\,\phi_4 \tag{11}$$

Where $N_1$, $N_2$, $N_3$, $N_4$ are the shapes functions and $\phi_1, \phi_2, \phi_3, \phi_4$ are the nodal variables. The values of the shape functions are

$$N_1 = \frac{(x-x_2)(y-y_4)}{(x_1-x_2)(y_1-y_4)},\ N_2 = \frac{(x-x_1)(y-y_3)}{(x_2-x_1)(y_2-y_3)},$$
$$N_3 = \frac{(x-x_4)(y-y_2)}{(x_3-x_4)(y_3-y_2)},\ N_4 = \frac{(x-x_3)(y-y_1)}{(x_4-x_3)(y_4-y_1)} \tag{12}$$

### 4.1) Evolution

In evolution, the finite element analysis and Lagrange method of multipliers approaches are used. The eq. (12) is solved by applying the finite element methodandafter that, the Lagrange method of multipliers approach is used to calculate the shortest distance from the interface by using the distance formula. The distance formula is written as follows

$$d = \sqrt{(x_i - x_I)^2 + (y_i - y_I)^2} \tag{13}$$

Where *d* is the distance, $x_i$ and $y_i$ are the coordinates of the nodes and $x_I$ and $y_I$ are the points on the interface ($\phi(x, y) = 0$).

### 4.2) Working

Consider the rectangular element having coordinates $\underline{X}_1, \underline{X}_2, \underline{X}_3, \underline{X}_4$ with their respective shape functions $N_1, N_2, N_3, N_4$ with Level-Set values $\phi_1, \phi_2, \phi_3, \phi_4$ as depicted in the given **Figure 3**

**Figure 3** – Schematic diagram of all the nodes

It is assumed that the $\phi_1$ for the node $\underline{X}_1$, $\phi_2$ for the node $\underline{X}_2$, $\phi_3$ for the node $\underline{X}_3$, $\phi_4$ for the node $\underline{X}_4$ and so on. Fit the polynomial over the rectangular element and calculate the shortest or minimum distance from the interface where it is equal to zero i.e. $\phi(x,y)=0$. The Lagrange method of multiplier approach is used to calculate the shortest or minimum distance given that it minimizes the function such that $\phi(x_I,y_I)=0$.

### 4.3) Lagrange Method of Multipliers and its Derivation

In mathematical optimization, the langrage method of the multipliers approach is used for minimizing or maximizing the function that is subjected to equality constraints. For our illustration, let us consider the optimization problem

Minimize $d(x,y)$

Subject to the condition $\phi(x,y)=0$

Where $d(x,y)$ shows the distance function and $\phi(x,y)$ is the Level-Set function. It is supposed that the first partial derivatives of both functions $d(x,y)$ and $\phi(x,y)$ are continuous. Now the new variable is introduced namely $(\lambda)$ and it is known as the Lagrange multiplier and its study is exemplified as follows

$$\mathcal{L}(x,y,\lambda)=d(x,y)+\lambda\cdot\phi(x,y) \quad (14)$$

Where the term $\lambda$ can be added or subtracted in eq. (14)

**Derivation**

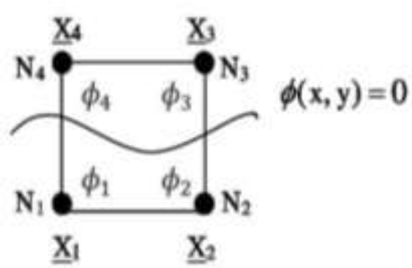
$$L(x_c,y_c)=d(x_c,y_c)+\lambda\,\phi(x_c,y_c)$$

Where, $S=d^2=(x_c-x_i)^2+(y_c-y_i)^2$ and $\phi(x_c,y_c)=a\,x_c+b\,x_cy_c+cy_c+d=0$

$x_c$ and $y_c$ are the unknown points and $x_i$ and $y_i$ are the known points respectively

$$\phi(x_c,y_c)=a\,x_c+b\,x_cy_c+cy_c+d=0$$

Where, $a,b,c,d$ are the coefficients of the equation and they are constants that can be negative and positive.



$$\frac{\partial L}{\partial x_I} = \frac{\partial S}{\partial x_I} + \lambda \frac{\partial \phi}{\partial x_I}$$

$$\frac{\partial L}{\partial x_I} = 2(x_I - x_i) + \lambda(a + b y_I) = 0$$

After simplification it becomes,

$$2x_I + \lambda b y_I = 2x_i - \lambda a \qquad (15)$$

$$\frac{\partial L}{\partial y_I} = \frac{\partial S}{\partial y_I} + \lambda \frac{\partial \phi}{\partial y_I}$$

$$\frac{\partial L}{\partial y_I} = 2(y_I - y_i) + \lambda(c + b x_I) = 0$$

After simplification it becomes,

$$\lambda b x_I + 2y_I = 2y_i - \lambda c \qquad 16)$$

$$\frac{\partial L}{\partial \lambda} = \frac{\partial S}{\partial \lambda} + \lambda \frac{\partial \phi}{\partial \lambda}$$

$$\frac{\partial L}{\partial \lambda} = \phi(x_I, y_I) = a\,x_I + b\,x_I y_I + c\,y_I + d = 0$$

$$a\,x_I + b\,x_I y_I + c\,y_I + d = 0 \qquad (17)$$

Solving Eq. (15) and Eq. (16) we get,

$$\underline{N}(x_I, y_I) = \left( \frac{4x_i - 2\lambda b y_i - 2\lambda a + \lambda^2 bc}{(4-\lambda^2 b^2)}, \frac{2x_i \lambda b - \lambda^2 ab - 4y_i + 2\lambda c}{-(4-\lambda^2 b^2)} \right)$$

Substitute the values of $x_I$ and $y_I$ in Eq. (17) we have,

$$a\,x_I + b\,x_I y_I + c\,y_I + d = 0$$

$$(db^4 - ab^3c)\lambda^4 + (4b^2cy_i + 12abc + 4b^3x_iy_i + 4b^2a\,x_i - 8db^2)\lambda^2$$
$$+(-8b^2x_i^2 - 8a^2 - 16x_ibc - 8c^2 - 16aby_i - 8b^2y_i^2)\lambda + 16a\,x_i + 16b\,x_iy_i + 16d + 16cy_i = 0$$

$$\underbrace{(db^4 - ab^3c)}_{A}\lambda^4 + \underbrace{0}_{B}\lambda^3 + \underbrace{(4b^2cy_i + 12abc + 4b^3x_iy_i + 4b^2a\,x_i - 8db^2)}_{C}\lambda^2$$
$$+\underbrace{(-8b^2x_i^2 - 8a^2 - 16x_ibc - 8c^2 - 16aby_i - 8b^2y_i^2)}_{D}\lambda + \underbrace{16a\,x_i + 16b\,x_iy_i + 16d + 16cy_i}_{E} = 0$$

Hence the final form is given as

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \qquad (18)$$

The equation (18) is the quadratic equation of $\lambda$ and the values of $\lambda$ will be used to compute the shortest / minimum distance from the interface to all the nodes of each element.

The reasoning behind the complex of roots is according to the fundamental theorem of algebra which states that "*the polynomial of degree one and its greatest powers contains at least one roots of the complex number system*" but in this research, the discriminant of the quadratic equation is less than zero and its physical significance according to this research is that the values of the $x_I$ and $y_I$ are computed by using the real roots which are lying at the interface and calculated the shortest distance from the point (with coordinates $x_I$ and $y_I$) of the interface to all the nodes and neglects the complex roots but after plugging the complex roots ($\lambda$) in the $x_I$ and $y_I$ relation so, it will give the complex values of $x_I$ and $y_I$ that would not be lying at the interface (Maybe it lies outside the interface). So, in this scenario, it is impossible to compute the shortest / minimum distance from that point (interface point) to all the nodes of the grid elements.

***Remark:*** It is remembered that if all the roots of the equation of the polynomial are complex, in this condition the shortest or minimum distance *d* is replaced by the actual Level-Set values ($\phi$)

**4.4) Plotting of the Level-Set Field before and after Re-initialization**

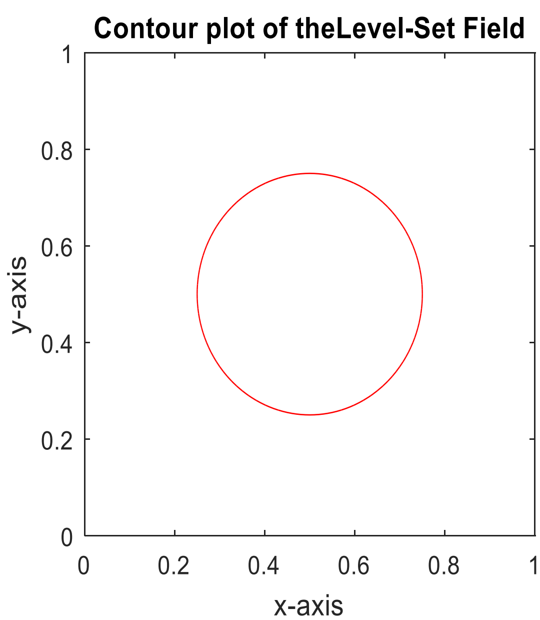



**Figure 3:** Actual Level-Set Field with its contours before re-initialization

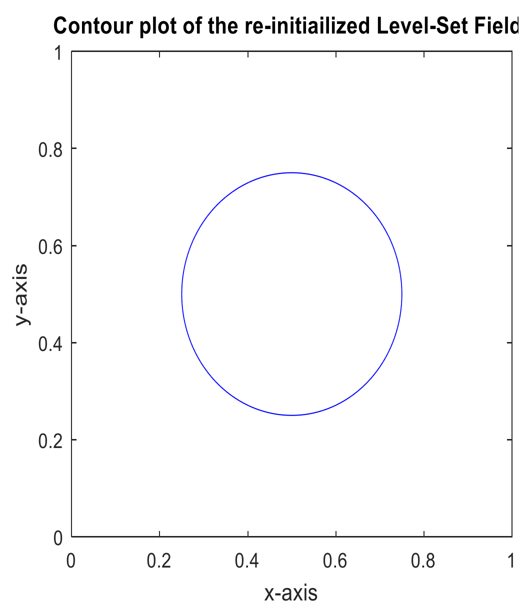



**Figure 4:** Actual Level-Set Field with its contours after re-initialization

**4.5) Errors and Norms of the re-initialized Level-Set Field**

| Mesh width (h) | Error Norm 2 | Order of Convergence | Error Norm infinity | Order of Convergence |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| $h = 0.01$ | 2.1323 | ----- | 0.1570 | ----- |
| $h = 0.005$ | 3.9938 | 0.5339 | 0.1827 | $0.8593 \approx 1$ |
| $h = 0.0025$ | 7.6450 | 0.5224 | 0.2034 | $0.8982 \approx 1$ |

**Table 1: Norm 2 and Norm infinity Errors of the re-initialized Level-Set Field with their Convergence**

## 5) TEST PROBLEMS

In this section, two problems are discussed namely

*(i) The Bubble Advection using a cross section of lens-shaped interface*

*(ii) Zalesak's rotating disc test*

### 5.1). The Bubble Advection with the Shape of lens-shaped Cross section.

At first, the well-known test case studied for the models of incompressible two-phase flow is a circular shape bubble. It depends on the liquid characteristics of materials and the ratio of the densities among the two phases, the variation in the contour of the bubble from its initial form to the final form.

Along with the domain $\Omega = [0,1] \times [0,1]$ the solution is attained, for the Level-Set (LS) field the initial condition is given as follows

$$\phi_l(\underline{X}, 0) = \max\{\phi_1(\underline{X}, 0), -\phi_2(\underline{X}, 0)\} \quad (19)$$

Where,

$$\phi_1(\underline{X}, 0) = \left| \underline{X} - \underline{X}_1^c(0) \right| - R \quad (20)$$

$$\phi_2(\underline{X}, 0) = \left| \underline{X} - \underline{X}_2^c(0) \right| - R \quad (21)$$

Here we have taken the values of $\underline{X}_1^c(0) = (0.5, 0.2)^T$, $\underline{X}_2^c(0) = (0.5, 0.35)^T$ and $R = 0.15$. so, before and after re-initialization the bubble advection with the lens-shaped interface is depicted in **Figure 6** and **Figure 7**.

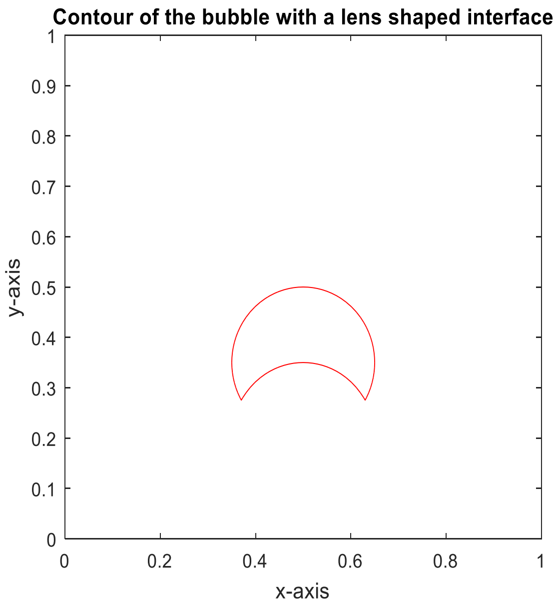



**Figure 5:** Bubble with a lens shape interface before re-initialization

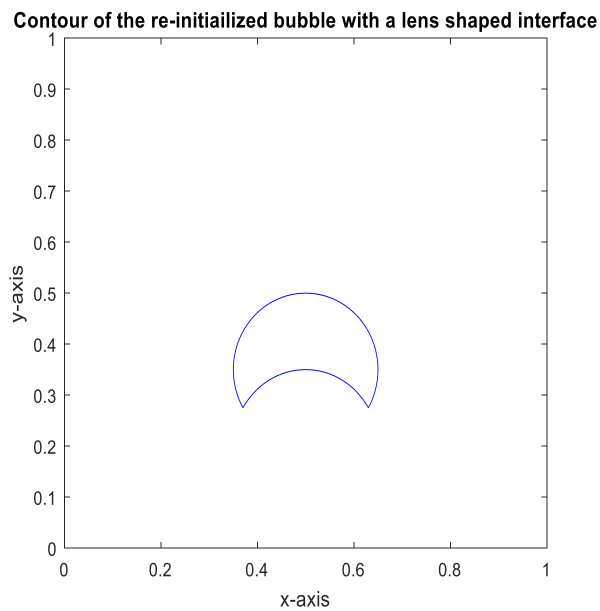



**Figure 6:** Bubble with a lens shape interface after re-initialization

### 5.1.1) Errors and Norms of the re-initialized bubble with a lens shape interface

| ***Mesh width (h)*** | **Error Norm 2** | **Order of Convergence** | **Error Norm infinity** | **Order of Convergence** |
|---|---|---|---|---|
| $h = 0.01$ | 4.2723 | ----- | 0.3766 | ----- |
| $h = 0.005$ | 7.5415 | $0.5665 \approx 1$ | 0.3901 | $0.9654 \approx 1$ |
| $h = 0.0025$ | 13.8561 | 0.5443 | 0.3965 | $0.9839 \approx 1$ |

**Table 2: Norm 2 and Norm infinity Errors of the re-initialized bubble with a lens shape interface with their convergence**

### 5.2). Zalesak's Rotating Disc

In the next case, Zalesak's rotating disc test problem has been discussed. It is defined over the same domain which is considered in the preceding test problem. For the Level-Set (LS) field the initial condition is given as follows

$$\phi_1(\underline{X},0)=\max\{\phi_1(\underline{X},0),-\phi_2(\underline{X},0)\} \tag{22}$$

Where,

$$\phi_1(\underline{X},0)=\left|\underline{X}-\underline{X}_1^c(0)\right|-R \tag{23}$$

$$\phi_2(\underline{X},0)=\max\left(\left|\underline{X}_1-\underline{X}_1^c(0)\right|-w,\left|\underline{X}_2-\underline{X}_2^c(0)+2w\right|-l\right) \tag{24}$$

Here we have taken the values of $\underline{X}^c(0)=(0.5,0.75)^T$, $R=0.15$and $\phi_2(\underline{X},0)$ represents the rectangular region with breath $w=R/6$and its length is $l=R$respectively. So, before and after re-initialization the Zalesak's rotating disc is depicted in **Figure 8** and **Figure 9**

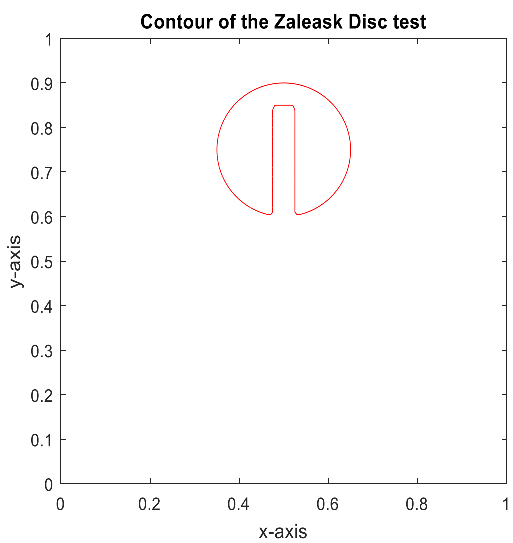



**Figure 7:** Zalesak's disc test before re-initialization

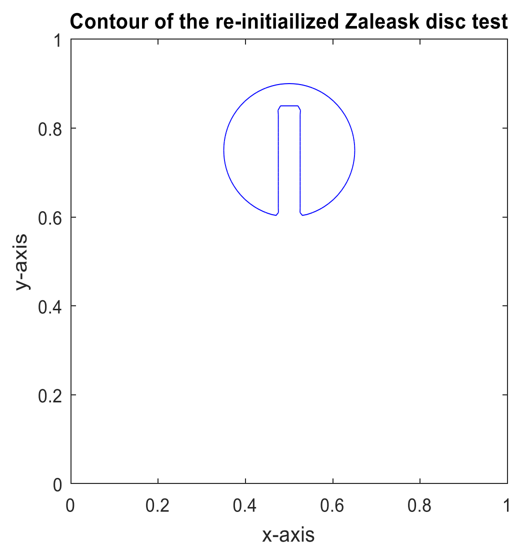



**Figure 8:** Zalesak's disc test after re-initialization

### 5.2.1) Errors and Norms of the re-initialized Zalesak's rotating disc

| Mesh width (h) | Error Norm 2 | Order of Convergence | Error Norm infinity | Order of Convergence |
|---|---|---|---|---|
| $h=0.01$ | 3.7053 | ----- | 0.2173 | ----- |
| $h=0.005$ | 7.1262 | 0.5200 | 0.2184 | $0.9950\approx 1$ |
| $h=0.0025$ | 13.9807 | 0.5097 | 0.2200 | $0.9927\approx 1$ |

**Table 3: Norm 2 and Norm infinity Errors of the re-initialized Zalesak's rotating disc with their convergence**

Zalesak's disc is a challenging problem due to its interface issue because in Zalesak's disc test problem interface is $C^0$continuous only and it has concave and convex both at the regions. The sharp corners of the interface are preserved by the Numerical techniques and are measured on their capability.

## 6) CONCLUSION

An efficient re-initialization method is presented for the Level-Set field based on signed distance computation. The proposed method is perfectly in-line in addition to the framework of the Finite Element Method suing discontinuous Galerkin Method (DGFEM). (i.e. element-wise operations). This method has the potential to extend it over higher-order polynomial approximation. Numerical test examples reveal the efficacy and effectiveness of the introduced scheme. In future research, the author may proceed with this research to the highest degree polynomials (i.e. 3rd degree, 4th degree, and so on) and implement this approach to the triangular and tetrahedral meshes.

## NOTATION

$\phi$ = Level-Set value / Level-Set function

$\Gamma(t)$ = Interface in time *t.*

$\Omega$ = Bounded region / computationally domain.

$\phi_t$ = Changing of interface with respect to time *t.*

$\underline{V}$ = Divergence-free velocity field.

$\nabla\phi$ = Gradient of the sign distance function.

$\Omega_k$ = Triangular volume control.

$L_i(x)$ = Legendre polynomial.

$n_p$ = Total number of nodal points.

$u_\alpha$ = Solenoid velocity.

$\hat{n}$ = Pointing outward normal.

$\phi_k^h$ = Numerical flux.

$M^k$ = Matrix of mass.

$S^k$ = Matrix of Stiffness.

$F^k$ = Element operator that operates boundary.

*N*= Shape function of the Lagrange polynomial.

$x_i$ and $y_i$ = Coordinates of the nodes.

$x_i$ and $y_i$ = Coordinates of the nodes.

$x_I$ and $y_I$ = Points on the interface.

$\lambda$ = Lagrange multiplier.

$T$ = Transpose of the matrix.

$R$ = Radius of the circle
$w$ = Breath of the rectangular region.
$l$ = Length of the rectangular region.

**REFERENCES**

[1] Gross, J.S. Hesthaven. "A Level-Set discontinuous Galerkin method for free surface flows", *Computational Methods Appl. Mech. Engrg* 2005. 195 (25-28), pp. 3406–3429.

[2] Stanley Osher, Jue Yan, "Discontinuous Galerkin Level-Set Method for Interface Capturing", November 2005, pp. 1 – 16.

[3] Daniel Hartmann, Matthias Meinke, Wolfgang Schroder, "Differential equation based constrained reinitialization for Level-Set methods", *Journal of Computational Physics* 227 (14), March 26, 2008, p. 6821–6845

[4] Roberto F. Ausas, Enzo A. Dari, Gustavo C. Buscaglia, "A Mass-Preserving Geometry-Based Reinitialization method for the Level-Set Function", *Asociación Argentina de Mecánica Computacional, Mecánica Computacional* Vol. 27, 10-13 Noviembre, 2008, pp. 13 – 32.

[5] Fahim Raees, Duncan Roy van der Heul, C. Vuik, "A mass-conserving Level-Set method", *International Journal for Numerical Methods in Fluids* · October 2015.

[6] Fahim Raees, Duncan Roy van der Heul, C. Vuik. "Evaluation of the interface-capturing algorithm of Open Foam for the simulation of incompressible immiscible two-phase flow", September 19, 2011, pp. 1 – 41.

[7] Gingold, R. A., & Monaghan, J. J, "Smoothed particle hydrodynamics: theory and application to non-spherical stars", Monthly Notices of the royal astronomical society, 181(3), 1977. pp. 375-389.

[8] Chen, S., & Doolen, G. D., "Lattice Boltzmann method for fluid flows", *Annual Review of Fluid Mechanics* 1998, 30(1), pp. 329-364.

[9] Gremaud, P. A., Kuster, C. M., & Li, Z., "A study of numerical methods for the Level-Set approach", *Applied Numerical Mathematics* 2007, 57(5-7), pp. 837-846.

[10] Naber, J., & Koren, B., "A Runge-Kutta discontinuous-Galerkin Level-Set method for unsteady compressible two-fluid flow". 2007.

[11] Hesthaven, J. S., & Warburton, T., "Discontinuous Galerkin methods for the time-domain Maxwell" s equations", *ACES Newsletter*, *19*(EPFL-ARTICLE-190449), 2004. pp. 10-29.

[12] Owkes, M., & Desjardins, O., "A discontinuous Galerkin conservative Level-Set scheme for interface capturing in multiphase flows". *Journal of Computational Physics* 2013, 249, pp. 275-302.

[13] Sethian, J. A., & Vladimirsky, "A. Fast methods for the Eikonal and related Hamilton–Jacobi equations on unstructured meshes". Proceedings of the National Academy of Sciences 2000, 97(11), pp. 5699-5703.

[14] Sethian, J. A., & Vladimirsky, "A Ordered upwind methods for static Hamilton--Jacobi equations", Theory and algorithms, *SIAM Journal on Numerical Analysis* 2003, 41(1), pp. 325-363.

[15] Gottlieb, S., & Shu, C. W., "Total variation diminishing Runge-Kutta schemes", *Mathematics of computation of the American Mathematical Society* 1998, 67(221), pp. 73-85.

[16] Jiang, G. S., & Peng, D., "Weighted ENO schemes for Hamilton--Jacobi equations", *SIAM Journal on Scientific computing* 2000, 21(6), pp. 2126-2143.

[17] Osher, S., & Sethian, J. A., "Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations", *Journal of Computational Physics* 1988, 79(1), pp.12-49.

[18] W. Rider and D. Kothe., "A marker particle method for interface tracking", Proceedings of the Sixth International Symposium on Computational Fluid Dynamics 1995, pp. 976-981

[19] Hirt, C. W., & Nichols, B. D., "Volume of fluid (VOF) method for the dynamics of free boundaries". *Journal of computational physics* 1981, 39(1), pp. 201-225.

[20] Cockburn, B., & Shu, C. W., "The Runge–Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems", *Journal of Computational Physics* 1998, 141(2), pp. 199-224.

[21] Raees F., "A Mass-Conserving hybrid interface capturing methods for the geometrically complicated domain", PhD thesis. TU Delft University of Technology, The Netherlands, 2016.

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