Morphisms in Trilogarithmic Tangent Complex of Order Three

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Abstract- In this paper we present maps and other necessary ingredients in third order trilogarithmic tangent groups and use these maps to relate the Grassmannian complex to Tangential trilogarithmic complex of order three. We also extend seigel's cross ratio and five term relation to order 3 and finally prove the commutativity of resulting diagrams. This work will play a key role in the generalization of these constructions.

Index Terms: Cutting, Greenhouse, rate, Substrate, Survival, Rosemary.

I. INTRODUCTION

The Grassmannian Complex is for the first time structured by A. Suslin which is used as a major tool to study co-homology of motivic complexes, knot theory and theory of manifolds (See [5], [9]). The natural connectivity between Grassmannian complex and polylogarithmic groups is one of its key feature which always captivated the renowned researchers to study since its beginning. In [3] the renowned mathematician A.B. Gonchrove derived motivic complexes in order to verify the Conjecture of Zagier on polylogs. Later, Cathelineau classified motivic complexes into two classes i.e. tangential complex and infinitesimal complex. Moreover, he produced tangential groups on dual numbers and used them to formulate tangential complex (See [1] and [2]). Another researcher who worked on these complexes is Siddiqui who derived the cross-ratio, functional equations of tangential complex and also formed certain morphisms. He used his constructions to link the Grassmannian complex and Tangential complex for the order one(See [10], [11] and [12],[13],[14]).

In [5] we broadened the constructions of Siddiqui for the second order. We also formed certain maps π^3_{0,ε^n} and π^3_{1,ε^n} in order to generalize the connection between Grasmmanian complex and Tangential complex for weight 2. However, the current study deals with the study of maps in the third order of Trilogarithmic tangential complexes as we are naturally motivated to extend these constructions up to a general order. To do this the study of only first two orders are not enough as they do not reflect any specific pattern. So the study of next order is essential after which we can attain the required goal. In the first section we introduce the group $T\mathcal{B}_3^3(F)$ as third ordered tangent group in weight 3. This group also satisfies functional equations of trilogarithms. After that we produce cross ratio, triple-ratio and seigel's identity and then we give a map ∂_{ε^3} in equation (13) to construct trilogarithmic tangential complex. Moreover we propose three more maps π^3_{0,ε^3}, π^3_{1,ε^3} and π^3_{2,ε^3} which enable us to formulate the diagram (B). In the last section we demonstrate proof of the commutativity of left and right squares of the diagram (B)).

I. MATERIALS AND METHODS

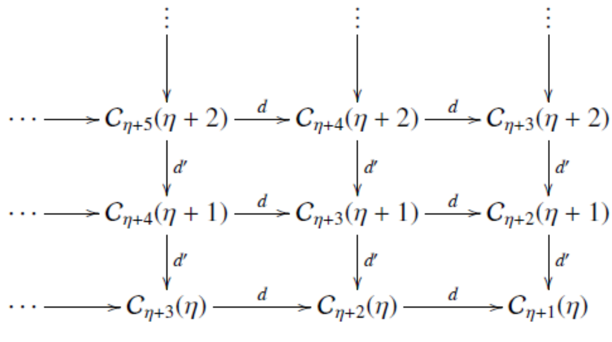
2.1 Grassmannian Complex

Let us consider $\mathcal{C}_m(\eta)$ be a free commutative group generated by the configurations $(\ell_1, \ldots, \hat{\ell}_i, \ldots, \ell_m)$ of the elements of an η-dimensional vector space V_η and $(\ell_i|\ell_1, \ldots, \hat{\ell}_i, \ldots, \ell_m)$ be the projective configuration of the vectors ℓ_j along the vectors ℓ_i, where $i \neq j$; $j = 0, \ldots, m$, then we can define differential maps $d: \mathcal{C}_{(m+1)}(\eta) \to \mathcal{C}_m(\eta)$ and $d': \mathcal{C}_{(m+1)}(\eta+1) \to \mathcal{C}_m(\eta)$ as

$$d: (\ell_1, \ldots, \ell_m) \mapsto \sum_{i=0}^{m} (-1)^m (\ell_1, \ldots, \hat{\ell}_i, \ldots, \ell_m)$$

$$d': (\ell_1, \ldots, \ell_m) \mapsto \sum_{i=0}^{m} (-1)^m (\ell_i|\ell_1, \ldots, \hat{\ell}_i, \ldots, \ell_m)$$

Above settings lead us to an algebraic complex given as



known as Grassmannian Complex.

2.2 Trilogarithmic Complexes of The Third Order Tangent Groups

Let $F[\varepsilon]_4$ be a truncated polynomial ring over an arbitrary field F then the $\mathbb{Z}$-module $T\mathcal{B}_2^3(F)$ is called a tangent group of order 3 if it is generated by $\langle s; s', s'', s'''] \in Z[F[\varepsilon]_4]$ and quotient by kernel of the map

$$\partial: Z[F[\varepsilon]_4] \longrightarrow (T\mathcal{B}_2^3(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F))$$

where ∂ can be described as

$$\partial(\langle s;t_1,t_2,t_3]_2) = \langle s;t_1,t_2,t_3]_2^3 \otimes s + \left(\frac{3t_3}{s} - \frac{3t_1t_2}{s^2} + \frac{t_1^3}{s^3}\right) \otimes [s]_2 \quad (1)$$

where $\langle s;t_1,t_2,t_3] = [s + t_1\varepsilon + t_2\varepsilon^2 + t_3\varepsilon^3] - [s], (s,t_1,t_2,t_3 \in F)$.

Use $T\mathcal{B}_3^3(F)$ to form the complex below

$$T\mathcal{B}_3^3(F) \xrightarrow{\partial_{\varepsilon^3}} (T\mathcal{B}_2^3(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) \xrightarrow{\partial_{\varepsilon^3}} \left(F \otimes \overset{2}{\wedge} F^\times\right) \oplus \left(\overset{3}{\wedge} F\right) \quad (2)$$

Our aim is to propose morphisms between this complex and the famous Grassmannian complex.

2.3 Cross-Ratio

In [5] cross ratio has been constructed for second order tangential settings so we use here the similar technique to extend it to third order. Let $A^3_{F[\varepsilon]_4}$ represent an affine space over the turncated ring of polynomials $F[\varepsilon]_4$ and $(v_0^*,\dots,v_3^*)$ belongs to $C_4(A^3_{F[\varepsilon]_4})$ then we obtain

$$r(v_0^*,\dots,v_3^*) = \frac{\{\sum_{j=0}^3 (v_0^*,v_3^*)_{\epsilon^j}\epsilon^j\}\{\sum_{j=0}^3 (v_1^*,v_2^*)_{\epsilon^j}\epsilon^j\}}{\{\sum_{j=0}^3 (v_0^*,v_2^*)_{\epsilon^j}\epsilon^j\}\{\sum_{j=0}^3 (v_1^*,v_3^*)_{\epsilon^j}\epsilon^j\}}.$$

From now we use a short hand $\mathbb{V}_{0n}^*$ to denote the tuple $(v_0^*,\dots,v_n^*)$ of $C_n(A^n_{F[\varepsilon]_{n+1}})$. Now we reform the above ratio into simpler form $G + H\epsilon + I\epsilon^2 + J\epsilon^3$, where the unknowns G , H and I represents $r(\mathbb{V}_{03}^*)$, $r_\varepsilon(\mathbb{V}_{03}^*)$ and $r_{\varepsilon^2}(\mathbb{V}_{03}^*)$ respectively. Values of first two of these coefficients are calulated in [10] and [5] respectively. In the same way we can evaluate "J" or $r_{\varepsilon^3}(\mathbb{V}_{03}^*)$ as follows

$$r_{\varepsilon^3}(\mathbb{V}_{03}^*) = \frac{\{(v_0^*,v_3^*)(v_1^*,v_2^*)\}_{\varepsilon^3}}{(v_0,v_2)(v_1,v_3)} - r_\varepsilon(\mathbb{V}_{03}^*)\frac{\{(v_0^*,v_2^*)(v_1^*,v_3^*)\}_{\varepsilon^2}}{(v_0,v_2)(v_1,v_3)} - r_{\varepsilon^2}(\mathbb{V}_{03}^*)\frac{\{(v_0^*,v_2^*)(v_1^*,v_3^*)\}_{\varepsilon}}{(v_0,l_2)(v_1,v_3)} - r(\mathbb{V}_{03})\frac{\{(v_0^*,v_2^*)(v_1^*,v_3^*)\}_{\varepsilon^3}}{(v_0,v_2)(v_1,v_3)} \quad (3)$$

where $\{(v_j^*,v_k^*)(v_m^*,v_n^*)\}_{\varepsilon^3} = \sum_{i=0}^3 (v_j^*,v_k^*)_{\varepsilon^{3-i}}(v_m^*,v_n^*)_{\varepsilon^i}$.

The triple-ratios $r_{3,\varepsilon^3}(\mathbb{V}_{05}^*)$ of six points over the truncated ring of polynomials $F[\varepsilon]_2$ and $F[\varepsilon]_3$ are described in [10] and [5] so we write here only for the ring $F[\varepsilon]_4$.

$$r_{3,\varepsilon^3}(\mathbb{V}_{05}^*) = \frac{\{(v_0^*v_1^*v_3^*)(v_1^*v_2^*v_4^*)(v_2^*v_0^*v_5^*)\}_{\varepsilon^3}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} - r_{3,\varepsilon}(\mathbb{V}_{05}^*)\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_{\varepsilon^2}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} - r_{3,\varepsilon^2}(\mathbb{V}_{05}^*)\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_{\varepsilon}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} - r_3(\mathbb{V}_{05})\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_{\varepsilon^3}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} \quad (4)$$

Here $(pqr)_{\varepsilon^3}$ represent the expression below

$$(pqr)_{\varepsilon^3} := p_\varepsilon q_{\varepsilon^0} r_{\varepsilon^3} + p_{\varepsilon^0} q_{\varepsilon^3} r_{\varepsilon^0} + p_{\varepsilon^3} q_{\varepsilon^0} r_{\varepsilon^0} + p_\varepsilon q_\varepsilon r_{\varepsilon^2} + p_{\varepsilon^0} q_{\varepsilon^2} r_\varepsilon + p_\varepsilon q_{\varepsilon^0} r_{\varepsilon^2} + p_\varepsilon q_\varepsilon r_\varepsilon + p_\varepsilon q_{\varepsilon^2} r_{\varepsilon^0} + p_{\varepsilon^2} q_{\varepsilon^0} r_\varepsilon + p_{\varepsilon^2} q_{\varepsilon^0} r_{\varepsilon^0} \quad (5)$$

Lemma 2.1 *If* $(\mathbb{V}_{03}^*)$ *is an element of* $C_4(A^3_{F[\varepsilon]_4})$ *then*

$$\{\Delta(v_0^*,v_1^*)\Delta(v_2^*,v_3^*)\}_{\varepsilon^3} = \{\Delta(v_0^*,v_2^*)\Delta(v_1^*,v_3^*)\}_{\varepsilon^3} - \{\Delta(v_0^*,v_3^*)\Delta(v_1^*,v_2^*)\}_{\varepsilon^3}$$

with $v_i^* = \sum_{k=0}^3 v_{i,\varepsilon^k}\varepsilon^k$; $v_{i,\varepsilon^0} = v_i$; $\Delta(v_i^*,v_j^*) = \sum_{k=0}^3 (v_i^*,v_j^*)_{\varepsilon^k}\varepsilon^k$ and $\Delta(v_i^*,v_j^*)_{\varepsilon^3} = \Delta(v_i,v_{j,\varepsilon^3}) + \Delta(l_{v,\varepsilon},v_{j,\varepsilon^2}) + \Delta(v_{i,\varepsilon^2},v_{j,\varepsilon}) + \Delta(v_{i,\varepsilon^3},v_j)$

Proof. We can deduce the proof directly from the result of Lemma 2.1 of [12] by setting = 3 .

II. RESULTS AND DISCUSSION

Effect of substrates nature on rosemary growth

Consider the following diagram

$$\begin{array}{ccccc} C_6(A^3_{F[\varepsilon]_4}) & \xrightarrow{d} & C_5(A^3_{F[\varepsilon]_4}) & \xrightarrow{d} & C_4(A^3_{F[\varepsilon]_4}) \\ \downarrow \pi^3_{2,\varepsilon^3} & & \downarrow \pi^3_{1,\varepsilon^3} & & \downarrow \pi^3_{0,\varepsilon^3} \\ T\mathcal{B}_3^3(F) & \xrightarrow{\partial_{\varepsilon^3}} & (T\mathcal{B}_2^3(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) & \xrightarrow{\partial_{\varepsilon^3}} & \left(F \otimes \overset{2}{\wedge} F^\times\right) \oplus \left(\overset{3}{\wedge} F\right) \end{array}$$

where

$$\pi^3_{0,\varepsilon^3}(\mathbb{V}_{03}^*) = \sum_{i=0}^3 (-1)^i \Bigg[\Bigg(3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_i,\dots,v_3)} - 3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_\varepsilon(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_i,\dots,v_3)^2} + \frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_\varepsilon^3}{(v_0,\dots,\hat{v}_i,\dots,v_3)^3}\Bigg) \otimes \frac{(v_0,\dots,\hat{v}_{i+1},\dots,v_3)}{(v_0,\dots,\hat{v}_{i+2},\dots,v_3)} \wedge \frac{(v_0,\dots,\hat{v}_{i+3},\dots,v_3)}{(v_0,\dots,\hat{v}_{i+2},\dots,v_3)} + \bigwedge_{\substack{j=0\\ j\neq i}}^3 \Bigg\{3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_i,\dots,v_3)} - 3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_\varepsilon(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_i,\dots,v_3)^2} + \frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_\varepsilon^3}{(v_0,\dots,\hat{v}_i,\dots,v_3)^3}\Bigg\}\Bigg], \quad i \bmod 4, \quad (6)$$

$$\pi^3_{1,\varepsilon^3}(\mathbb{V}_{04}^*) = -\frac{1}{3}\sum_{i=0}^4 (-1)^i \Big(\langle r(v_i|v_0,\dots,\hat{v}_i,\dots,v_4); r_\varepsilon(v_i^*|v_0^*,\dots,\hat{v}^*_i,\dots,v_4^*),\dots,r_{\varepsilon^3}(v_i^*|v_0^*,\dots,\hat{v}^*_i,\dots,v_4^*)]_2^3 \otimes \prod_{i\neq j}(\hat{v}_i,\hat{v}_j) + \sum_{\substack{j=0\\ j\neq i}}^4 \Big\{3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)} - 3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_\varepsilon(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)^2} + \frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_\varepsilon^3}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)^3}\Big\} \otimes [r(v_i|v_0,\dots,\hat{v}_i,\dots,v_4)]_2\Big) \quad (7)$$

and

$$\pi^3_{2,\varepsilon^3}(\mathbb{V}_{05}^*) = \frac{2}{45}\mathrm{Alt}_6\langle r_3(\mathbb{V}_{05}); r_{3,\varepsilon}(\mathbb{V}_{05}^*), r_{3,\varepsilon^2}(\mathbb{V}_{05}^*), r_{3,\varepsilon^3}(\mathbb{V}_{05}^*)]_3^3 \quad (8)$$

where

$$r_3(\mathbb{V}_{05}) = \frac{(v_0v_1v_3)(v_1v_2v_4)(v_2v_0v_5)}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} \quad (9)$$

and

$$r_{3,\varepsilon}(\mathbb{V}_{05}^*) = \frac{\{(v_0^*v_1^*v_3^*)(v_1^*v_2^*v_4^*)(v_2^*v_0^*v_5^*)\}_\varepsilon}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} - r_3(\mathbb{V}_{05})\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_\varepsilon}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} \quad (10)$$

$$r_{3,\varepsilon^2}(\mathbb{V}_{05}^*) = \frac{\{(v_0^*v_1^*v_3^*)(v_1^*v_2^*v_4^*)(v_2^*v_0^*v_5^*)\}_{\varepsilon^2}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} - r_{3,\varepsilon}(\mathbb{V}_{05}^*)\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_\varepsilon}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)}$$

$$-r_3(\mathbb{V}_{05})\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_{\varepsilon^2}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} \quad (11)$$

$$r_{3,\varepsilon^3}(\mathbb{V}_{05}^*) = \frac{\{(v_0^*v_1^*v_3^*)(v_1^*v_2^*v_4^*)(v_2^*v_0^*v_5^*)\}_{\varepsilon^3}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} - r_{3,\varepsilon}(\mathbb{V}_{05})\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_{\varepsilon^2}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)}$$

$$-r_{3,\varepsilon^2}(\mathbb{V}_{05}^*)\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_{\varepsilon}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} - r_3(\mathbb{V}_{05})\frac{\{(v_0^*v_1^*v_4^*)(v_1^*v_2^*v_5^*)(v_2^*v_0^*v_3^*)\}_{\varepsilon^3}}{(v_0v_1v_4)(v_1v_2v_5)(v_2v_0v_3)} \quad (12)$$

Now let us use the shorthand $s = r_3(\mathbb{V}_{05})$, $t_1 = r_{3,\varepsilon}(\mathbb{V}_{05}^*)$, $t_2 = r_{3,\varepsilon^2}(\mathbb{V}_{05}^*)$ and $t_3 = r_{3,\varepsilon^3}(\mathbb{V}_{05}^*)$ in order to exhibit the map ∂_{ε^3} as

$$\partial_{\varepsilon^3}(\langle s;t_1,t_2,t_3]_2^3 \otimes c + x\otimes[y]_2) = \delta^1+\delta^2 \quad (13)$$

where

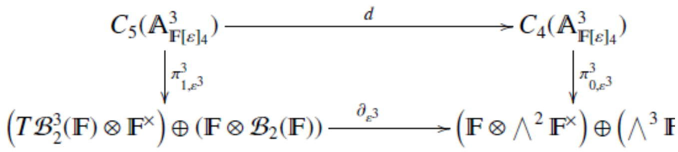
$$\delta^1(\langle s;t_1,t_2,t_3]_2^3 \otimes c + x\otimes[y]_2) = \left(\frac{3t_3}{s}-\frac{3t_1t_2}{s^2}+\frac{t_1^3}{s^3}\right)\otimes(1-s)\wedge c - \left(\frac{3t_3}{1-s}-\frac{3t_1t_2}{(1-s)^2}+\frac{t_1^3}{(1-s)^3}\right)\otimes s\wedge c + x\otimes(1-y)\wedge y \quad (14)$$

$$\delta^2(\langle s;t_1,t_2,t_3]_2^3 \otimes c + x\otimes[y]_2) = \left(\frac{3t_3}{s}-\frac{3t_1t_2}{s^2}+\frac{t_1^3}{s^3}\right)\wedge\left(\frac{3t_3}{1-s}-\frac{3t_1t_2}{(1-s)^2}+\frac{t_1^3}{(1-s)^3}\right)\wedge x \quad (15)$$

and

$$\partial_{\varepsilon^3}(\langle s;t_1,t_2,t_3]_3^3) = \langle s;t_1,t_2,t_3]_2^3\otimes s + \left(\frac{3t_3}{s}-\frac{3t_1t_2}{s^2}+\frac{t_1^3}{s^3}\right)\otimes[s]_2 \quad (16)$$

Theorem 2.2 *Commutation holds in the diagram given below*



i.e. $\partial_{\varepsilon^3}\circ\pi^3_{1,\varepsilon^3} = \pi^3_{0,\varepsilon^3}\circ d$

Proof. Definition of π^3_{0,ε^3} is given in (6) which seems in a complex form so we divide it as $\pi^3_{0,\varepsilon^3} = \phi_1+\phi_2$. Now we compute both parts $\phi_2\circ d(\mathbb{V}_{04}^*)$ and $\phi_1\circ d(\mathbb{V}_{04}^*)$ separately.

$$\phi_1\circ d(\mathbb{V}_{04}^*) = \phi_1\left(\sum_{i=0}^{4}(-1)^i(v_0^*,\dots,\hat{v}_i^*,\dots,v_4^*)\right)$$

$$= \widetilde{\mathrm{Alt}}_{(01234)}\Big[\sum_{i=0}^{3}(-1)^i\{3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_i,\dots,v_3)} - (3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon}(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_i,\dots,v_3)^2} - \frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon}^3}{(v_0,\dots,\hat{v}_i,\dots,v_3)^3})\otimes\frac{(v_0,\dots,\hat{v}_{i+1},\dots,v_3)}{(v_0,\dots,\hat{v}_{i+2},\dots,v_3)}\wedge\frac{(v_0,\dots,\hat{v}_{i+3},\dots,v_3)}{\Delta(v_0,\dots,\hat{v}_{i+2},\dots,v_3)}),\quad i \bmod 4\}\Big] \quad (17)$$

where $\widetilde{\mathrm{Alt}}_{(01234)}$ describes alternating sum. The simplification of internal sum provides us 12 terms of the form $p\otimes\frac{q}{r}\wedge\frac{s}{r}$ and by utilizing the relations $p\wedge\frac{q}{r} = p\wedge q - p\wedge r$ and $(p+q)\otimes r = p\otimes r + q\otimes r$, the terms become 48 which are of the form $p\otimes q\wedge r$. Then we apply alternating sum which gives us 240 terms. All these terms will be of three different types like $\frac{p_\varepsilon^3}{p^3}\otimes q\wedge r$, $3\frac{p_{\varepsilon^3}}{p}\otimes q\wedge r$ and $3\frac{p_\varepsilon p_{\varepsilon^2}}{p}\otimes q\wedge r$. Moreover we write the terms together whose first factor. For instance, the combined expression of the terms with common factor $\frac{(v_4^*,v_2^*,v_3^*)_\varepsilon^3}{\Delta(v_4,v_2,v_3)}\otimes\cdots$ is

$$\frac{(v_4^*,v_2^*,v_3^*)_\varepsilon^3}{\Delta(v_4,v_2,v_3)}\otimes(\Delta(v_0,v_3,v_4)\wedge(v_0,v_2,v_3)-(v_0,v_3,v_4)\wedge(v_0,v_2,v_4)-\Delta(v_0,v_2,v_4)\wedge(v_0,v_2,v_3)+\Delta(v_1,v_3,v_4)\wedge(v_1,v_2,v_4)-(v_1,v_3,v_4)\wedge(v_1,v_2,v_3)+(v_1,v_2,v_4)\wedge(v_1,v_2,v_3)) \quad (18)$$

and the terms having common factor $\frac{(v_1^*,v_2^*,v_3^*)_{\varepsilon^3}}{(v_1,v_2,v_3)}\otimes\cdots$ will be

$$\frac{(v_1^*,v_2^*,v_3^*)_{\varepsilon^3}}{(v_1,v_2,v_3)}\otimes((v_0,v_2,v_3)\wedge(v_0,v_1,v_2)-(v_0,v_2,v_3)\wedge(v_0,v_1,v_3-(v_0,v_1,v_3)\wedge(v_0,v_1,v_2)+(v_2,v_3,v_4)\wedge(v_1,v_3,v_4)-(v_2,v_3,v_4)\wedge(v_1,v_2,v_4)+(v_1,v_3,v_4)\wedge(v_1,v_2,v_4))$$

This concludes the calculation of $\phi_1\circ d(\mathbb{V}_{04}^*)$.

Other part $\phi_2\circ d(\mathbb{V}_{05}^*)$ can be evaluated as

$$= \widetilde{\mathrm{Alt}}_{(01234)}\{\bigwedge_{\substack{j=0\\j\neq i}}^{3}(3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_i,\dots,v_3)} - \left(3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon}(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_i,\dots,v_3)^2} - \frac{(v_0^*,\dots,\hat{v}_i^*,\dots,v_3^*)_{\varepsilon}^3}{(v_0,\dots,\hat{v}_i,\dots,v_3)^3}\right),\quad i \bmod 4\} \quad (19)$$

Let us move to calculate $\partial_{\varepsilon^3}\circ\pi^3_{1,\varepsilon^3}$. Using the definitions (7) and (13) we will have $\partial_{\varepsilon^3}\circ\pi^3_{1,\varepsilon^3}(\mathbb{V}_{04}^*)$

$$= \partial_{\varepsilon^3}(-\frac{1}{3}\sum_{i=0}^{4}(-1)^i(\langle r(v_i|v_0,\dots,\hat{v}_i,\dots,v_4); r_\varepsilon(v_i^*|v_0^*,\dots,\hat{v}^*{}_i,\dots,v_4^*),\dots,r_{\varepsilon^3}(v_i^*|v_0^*,\dots,\hat{v}^*{}_i,\dots,v_4^*)]_2^3 \otimes\prod_{i\neq j}(\hat{v}_i,\hat{v}_j) + \sum_{\substack{j=0\\j\neq i}}^{4}\{3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)} - 3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon}(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)^2} + \frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon}^3}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)^3}\}\otimes[r(v_i|v_0,\dots,\hat{v}_i,\dots,v_4)]_2)$$

Before the execution of ∂_{ε^3} we break it as

$$\partial_{\varepsilon^3} = \delta^1+\delta^2$$

Here we adopt the substitutions below in order to make our calculations easy $a_i = r(v_i|v_0,\dots,\hat{v}_i,\dots,v_4)$; $b_{i1} = r_\varepsilon(v_i^*|v_0^*,\dots,\hat{v}^*{}_i,\dots,v_4^*)$; $b_{i2} = r_{\varepsilon^2}(v_i^*|v_0^*,\dots,\hat{v}^*{}_i,\dots,v_4^*)$; $b_{i3} = r_{\varepsilon^3}(v_i^*|v_0^*,\dots,\hat{v}^*{}_i,\dots,v_4^*)$; $c_i = \prod_{i\neq j}\Delta(\hat{v}_i,\hat{v}_j)$; $y_i = [r(v_i|v_0,\dots,\hat{v}_i,\dots,v_4)]_2$ $x_i = \sum_{\substack{j=0\\j\neq i}}^{4}\{3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)} -$

$$3\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_\varepsilon(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)^2}+\frac{(v_0^*,\dots,\hat{v}_i^*,\dots,\hat{v}_j^*,\dots,v_4^*)_\varepsilon^3}{(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_4)^3}\} \tag{20}$$

The execution of δ^1 implies

$$\delta^1\circ\pi^3_{1,\varepsilon^3}(\mathbb{V}^*_{04}) = -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\{\left(\frac{3b_{3i}}{a_i}-\frac{3b_{1i}b_{2i}}{a_i^2}+\frac{b_{1i}^3}{a_i^3}\right)\otimes(1-a_i)\wedge c_i - \left(\frac{3b_{3i}}{1-a_i}-\frac{3b_{1i}b_{2i}}{(1-a_i)^2}+\frac{b_{1i}^3}{(1-a_i)^3}\right)\otimes a_i\wedge c_i + x\otimes(1-y_i)\wedge y_i\} \tag{21}$$

The expansion of summation for $i=0,\dots,4$ results in five different summands $S_i;\ i=0,\dots,4$ which we calculate separately

$$\delta^1\circ\pi^3_{1,\varepsilon^3}(\mathbb{V}^*_{04}) = S_0-S_1+S_2-S_3+S_4 \tag{22}$$

The value of I_4 can be evaluated by the substitution of $i=4$ in (21). This implies

$$S_4 = -\frac{1}{3}\{\left(\frac{3b_{34}}{a_4}-\frac{3b_{14}b_{24}}{a_4^2}+\frac{b_{14}^3}{a_4^3}\right)\otimes(1-a_4)\wedge c_4 - \left(\frac{3b_{34}}{1-a_4}-\frac{3b_{14}b_{24}}{(1-a_4)^2}+\frac{b_{14}^3}{(1-a_4)^3}\right)\otimes a_4\wedge c_4 + x\otimes(1-y_4)\wedge y_4\} \tag{23}$$

where $a_4 = r(v_4|v_0,\dots,v_3)$; $b_{14} = r_\varepsilon(v_4^*|v_0^*,\dots,v_3^*)$; $b_{24} = r_{\varepsilon^2}(v_4^*|v_0^*,\dots,v_3^*)$; $b_{34} = r_{\varepsilon^3}(v_4^*|v_0^*,\dots,v_3^*)$; $y_4 = [r(v_4|v_0,\dots,v_3)]_2$; $c_4 = \prod_{j=0}^{3}\Delta(\hat{v}_4,\hat{v}_j)$.

$$x_4 = \sum_{j=0}^{3}\{3\frac{(v_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_j,\dots,v_3)} - 3\frac{(v_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_\varepsilon(v_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_j,\dots,v_3)^2} + \frac{(v_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_\varepsilon^3}{(v_0,\dots,\hat{v}_j,\dots,v_3)^3}\} \tag{24}$$

Here we find the values of $\frac{3b_{34}}{a_4}-\frac{3b_{14}b_{24}}{a_4^2}+\frac{b_{14}^3}{a_4^3}$ and $\frac{3b_{34}}{1-a_4}-\frac{3b_{14}b_{24}}{(1-a_4)^2}+\frac{b_{14}^3}{(1-a_4)^3}$.

$$\frac{b_{14}^3}{a_4^3} = \left(\frac{r_\epsilon(v_4^*|v_0^*,\dots,v_3^*)}{r(v_4|v_0,\dots,v_3)}\right)^3$$
$$= \frac{(v_4^*v_0^*v_3^*)_\epsilon^3}{(v_4v_0v_3)^3}+\frac{(v_4^*v_1^*v_2^*)_\epsilon^3}{(v_4v_1v_2)^3}-\frac{(v_4^*v_0^*v_2^*)_\epsilon^3}{(v_4v_0v_2)^3}-\frac{(v_4^*v_1^*v_3^*)_\epsilon^3}{(v_4v_1v_3)^3}+$$
$$3\frac{(v_4^*v_0^*v_3^*)_\epsilon^2}{(v_4v_0v_3)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-3\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}$$
$$+3\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon^2}{(v_4v_1v_2)^2}-3\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}-$$
$$3\frac{(v_4^*v_0^*v_3^*)_\epsilon^2}{(v_4v_0v_3)^2}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}+3\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}-$$
$$3\frac{(v_4^*v_0^*v_3^*)_\epsilon^2}{(v_4v_0v_3)^2}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+3\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}-$$
$$3\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_2^*)_\epsilon^2}{(v_4v_1v_2)^2}+3\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-$$
$$3\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon^2}{(v_4v_1v_2)^2}+3\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-$$
$$6\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}-6\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}$$
$$\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+6\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}+$$
$$6\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)} \tag{25}$$

$$\frac{b_{14}b_{24}}{a_4^2} = \left(\frac{r_\epsilon(v_4^*|v_0^*,\dots,v_3^*)r_{\epsilon^2}(v_4^*|v_0^*,\dots,v_3^*)}{r(v_4^*|v_0,\dots,v_3)}\right)^2$$
$$= \frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}+\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}+$$
$$\frac{(v_4^*v_0^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}-\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}$$
$$\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}-\frac{(v_4^*v_0^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}-\frac{(v_4^*v_0^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}+$$
$$\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}+\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}$$
$$+\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}-\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}-$$
$$\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}-\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}-\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}$$
$$\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}+$$
$$\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+\frac{(v_4^*v_0^*v_3^*)_\epsilon^2}{(v_4v_0v_3)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-2\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}$$
$$+\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon^2}{(v_4v_1v_2)^2}-$$
$$2\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}-\frac{(v_4^*v_0^*v_3^*)_\epsilon^2}{(v_4v_0v_3)^2}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}+$$
$$2\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}-\frac{(v_4^*v_0^*v_3^*)_\epsilon^2}{(v_4v_0v_3)^2}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+$$
$$2\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}-\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_2^*)_\epsilon^2}{(v_4v_1v_2)^2}+$$
$$2\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon^2}{(v_4v_1v_2)^2}$$
$$+2\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-\frac{(v_4^*v_0^*v_2^*)_\epsilon^3}{(v_4v_0v_2)^3}-\frac{(v_4^*v_1^*v_3^*)_\epsilon^3}{(v_4v_1v_3)^3}-$$
$$3\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0l_2)}-3\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}$$
$$\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+3\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}+$$
$$3\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)} \tag{26}$$

$$\frac{b_{34}}{a_4} = \frac{(v_4^*v_0^*v_3^*)_{\epsilon^3}}{(v_4v_0v_3)}+\frac{(v_4^*v_1^*v_2^*)_{\epsilon^3}}{(v_4v_1v_2)}-\frac{(v_4^*v_0^*v_2^*)_{\epsilon^3}}{(v_4v_0v_2)}-\frac{(v_4^*v_1^*v_3^*)_{\epsilon^3}}{(v_4v_1v_3)}+$$
$$2\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}+2\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}$$
$$+\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}+\frac{(v_4^*v_0^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}-$$
$$\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}-\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}-\frac{(v_4^*v_0^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}$$
$$\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}-\frac{(v_4^*v_0^*v*_3)_{\epsilon^2}}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0l_2)}-\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}-$$
$$\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}-\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}$$
$$-\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}+$$
$$\frac{(v_4^*v_0^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}-\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}-\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}$$
$$\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}+\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}+\frac{(v_4^*v_0^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}+$$

$$\frac{(v_4^*v_o^*v_2^*)_\epsilon^2}{(v_4v_0v_2)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}+\frac{(v_4^*v_1^*v_3^*)_\epsilon^2}{(v_4v_1v_3)^2}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)} -\frac{(v_4^*v_0^*v_2^*)_\epsilon^3}{(v_4v_0v_2)^3}-\frac{(v_4^*v_1^*v_3^*)_\epsilon^3}{(v_4v_1v_3)^3}- \frac{(v_4^*v_o^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}-\frac{(v_4^*v_o^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}+ \frac{(v_4^*v_o^*v_3^*)_\epsilon}{(v_4v_0v_3)} \frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)}+\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_0^*v_2^*)_\epsilon}{(v_4v_0v_2)} \quad (27)$$

Using equations (25) , (26) and (27) we get

$$\frac{3b_{34}}{a_4}-\frac{3b_{14}b_{24}}{a_4^2}+\frac{b_{14}^3}{a_4^3} = 3(\frac{(v_4^*v_o^*v_3^*)_{\epsilon^3}}{(v_4v_0v_3)}+\frac{(v_4^*v_1^*v_2^*)_{\epsilon^3}}{(v_4v_1v_2)}-\frac{(v_4^*v_o^*v_2^*)_{\epsilon^3}}{(v_4v_0v_2)}-\frac{(v_4^*v_1^*v_3^*)_{\epsilon^3}}{(v_4v_1l_3)}+ \frac{(v_4^*v_o^*v_2^*)_\epsilon}{(v_4v_0v_2)}\frac{(v*_4v_o^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}+\frac{(v_4^*v_1^*v_3^*)_\epsilon}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)} -\frac{(v_4^*v_o^*v_3^*)_\epsilon}{(v_4v_0v_3)}\frac{(v_4^*v_o^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}-\frac{(v_4^*v_1^*v_2^*)_\epsilon}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)})+\frac{(v_4^*v_o^*v_3^*)_\epsilon^3}{(v_4v_0v_3)^3}+ \frac{(v_4^*v_1^*v_2^*)_\epsilon^3}{(v_4v_1v_2)^3}-\frac{(v_4^*v_0^*v_2^*)_\epsilon^3}{(v_4v_0v_2)^3}-\frac{(v_4^*v_1^*v_3^*)_\epsilon^3}{(v_4v_1v_3)^3} \quad (28)$$

Similarly,

$$\frac{3b_{34}}{1-a_4}-\frac{3b_{14}b_{24}}{(1-a_4)^2}+\frac{b_{14}^3}{(1-a_4)^3} = 3(\frac{(v_4^*v_0^*v_2^*)_{\varepsilon^3}}{(v_4v_0v_2)}+\frac{(v_4^*v_1^*v_3^*)_{\varepsilon^3}}{(v_4v_1v_3)}-\frac{(v_4^*v_0^*v_1^*)_{\varepsilon^3}}{(v_4v_0v_1)}-\frac{(v_4^*v_2^*v_3^*)_{\varepsilon^3}}{(v_4v_2v_3)}- \frac{(v_4^*v_0^*v_2^*)_{\varepsilon^2}}{(v_4v_0v_2)}\frac{(v_4^*v_0^*v_2^*)_{\varepsilon}}{(v_4v_0v_2)}-\frac{(v_4^*v_1^*v_3^*)_{\varepsilon^2}}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_3^*)_{\varepsilon}}{(v_4v_1v_3)}+ \frac{(v_4^*v_0^*v_1^*)_{\varepsilon^2}}{(v_4v_0v_1)}\frac{(v_4^*v_0^*v_1^*)_{\varepsilon}}{(v_4v_0v_1)}+\frac{(v_4^*v_2^*v_3^*)_{\varepsilon^2}}{(v_4v_2v_3)}\frac{(v_4^*v_2^*v_3^*)_{\varepsilon}}{(v_4v_2v_3)}) +\frac{(v_4^*v_0^*v_2^*)_\varepsilon^3}{(v_4v_0v_2)^3}+\frac{(v_4^*v_1^*v_3^*)_\varepsilon^3}{(v_4v_1v_3)^3}-\frac{(v_4^*v_0^*v_1^*)_\varepsilon^3}{(v_4v_0v_1)^3}-\frac{(v_4^*v_2^*v_3^*)_\varepsilon^3}{(v_4v_2v_3)^3} \quad (29)$$

Now equation (23) gives us the value of S_4

$$S_4 = -\frac{1}{3}\{(3\frac{(v_4^*v_o^*v_3^*)_{\epsilon^3}}{(v_4v_0v_3)}+3\frac{(v_4^*v_1^*v_2^*)_{\epsilon^3}}{(v_4v_1v_2)}- 3\frac{(v_4^*v_0^*v_2^*)_{\epsilon^3}}{(v_4v_0v_2)}-3\frac{(v_4^*v_1^*v_3^*)_{\epsilon^3}}{(v_4v_1v_3)}+3\frac{(v_4^*v_0^*v_2^*)_{\epsilon}}{(v_4v_0v_2)}\frac{(v_4^*v_o^*v_2^*)_{\epsilon^2}}{(v_4v_0v_2)}+ 3\frac{(v_4^*v_1^*v_3^*)_{\epsilon}}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_3^*)_{\epsilon^2}}{(v_4v_1v_3)}-3\frac{(v_4^*v_o^*v_3^*)_{\epsilon}}{(v_4v_0v_3)}\frac{(v_4^*v_o^*v_3^*)_{\epsilon^2}}{(v_4v_0v_3)}- 3\frac{(v_4^*v_1^*v_2^*)_{\epsilon}}{(v_4v_1v_2)}\frac{(v_4^*v_1^*v_2^*)_{\epsilon^2}}{(v_4v_1v_2)}+\frac{(v_4^*v_o^*v_3^*)_\epsilon^3}{(v_4v_0v_3)^3}+\frac{(v_4^*v_1^*v_2^*)_\epsilon^3}{(v_4v_1v_2)^3}- \frac{(v_4^*v_0^*v_2^*)_\epsilon^3}{(v_4v_0v_2)^3}-\frac{(v_4^*v_1^*v_3^*)_\epsilon^3}{(v_4v_1v_3)^3})\otimes\frac{(v_4v_0v_1)(v_4v_2v_3)}{(v_4v_0v_2)(v_4v_1v_3)}\wedge (v_1v_2v_3)(v_0v_2v_3)(v_0v_1v_3)(v_0v_1v_2)$$

$$-(3\frac{(v_4^*v_0^*v_2^*)_{\varepsilon^3}}{(v_4v_0v_2)}+3\frac{(v_4^*v_1^*v_3^*)_{\varepsilon^3}}{(v_4v_1v_3)}-3\frac{(v_4^*v_0^*v_1^*)_{\varepsilon^3}}{(v_4v_0v_1)}-3\frac{(v_4^*v_2^*v_3^*)_{\varepsilon^3}}{(v_4v_2v_3)}- 3\frac{(v_4^*v_0^*v_2^*)_{\varepsilon^2}}{(v_4v_0v_2)}\frac{(v_4^*v_0^*v_2^*)_{\varepsilon}}{(v_4v_0v_2)}- 3\frac{(v_4^*v_1^*v_3^*)_{\varepsilon^2}}{(v_4v_1v_3)}\frac{(v_4^*v_1^*v_3^*)_{\varepsilon}}{(v_4v_1v_3)}+3\frac{(v_4^*v_0^*v_1^*)_{\varepsilon^2}}{(v_4v_0v_1)}\frac{(v_4^*v_0^*v_1^*)_{\varepsilon}}{(v_4v_0v_1)}+ 3\frac{(v_4^*v_2^*v_3^*)_{\varepsilon^2}}{(v_4v_2v_3)}\frac{(v_4^*v_2^*v_3^*)_{\varepsilon}}{(v_4v_2v_3)}+\frac{(v_4^*v_0^*v_2^*)_\varepsilon^3}{(v_4v_0v_2)^3} +\frac{(v_4^*v_1^*v_3^*)_\varepsilon^3}{(v_4v_1v_3)^3}-\frac{(v_4^*v_0^*v_1^*)_\varepsilon^3}{(v_4v_0v_1)^3}-\frac{(v_4^*v_2^*v_3^*)_\varepsilon^3}{(v_4v_2v_3)^3})\otimes\frac{(v_4v_0v_3)(v_4v_2v_2)}{(v_4v_0v_2)(v_4v_1v_3)}\wedge (v_1v_2v_3)(v_0v_2v_3)(v_0v_1v_3)(v_0v_1v_2)+$$

$$\sum_{j=0}^{3}\{3\frac{(v_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_{\varepsilon^3}}{(v_0,\dots,\hat{v}_j,\dots,v_3)} -3\frac{(v_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_{\varepsilon}(v_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_{\varepsilon^2}}{(v_0,\dots,\hat{v}_j,\dots,v_3)^2} + \frac{(l_0^*,\dots,\hat{v}_j^*,\dots,v_3^*)_\varepsilon^3}{(v_0,\dots,\hat{v}_j,\dots,v_3)^3}\}\otimes(1-y_4)\wedge y_4\} \quad (27)$$

we use the relations $x\otimes\frac{y}{z}=x\otimes y-x\otimes z$ and $x\otimes yz=x\otimes y+x\otimes z$ for simplification which gives us a total of 576 terms. All of these terms will be of three different types $3\frac{(v_i^*,v_j^*,v_k^*)_{\varepsilon^3}}{(v_i,v_j,v_k)}\otimes l\wedge m$, $\frac{(v_i^*,v_j^*,v_k^*)_{\varepsilon^2}}{(v_i,v_j,v_k)}\frac{(v_i^*,v_j^*,v_k^*)_{\varepsilon}}{(v_i,v_j,v_k)}\otimes l\wedge m$ and $\frac{(v_i^*,v_j^*,v_k^*)_\varepsilon^3}{(v_i,v_j,v_k)^3}\otimes l\wedge m$ and will contain 192 terms each.

In a similar fashion we can determine S_0,S_1,S_2 and S_3 and then we substitute into (22) . We combine the terms (28) whose common factor factors are identical $\frac{\Delta(v_i^*,v_j^*,v_k^*)}{\Delta(v_i,v_j,v_k)}\otimes\cdots$. For example terms having common factor $\frac{\Delta(v_4^*,v_2^*,v_3^*)_\varepsilon^3}{\Delta(v_4,v_2,v_3)}\otimes\cdots$ make the expression

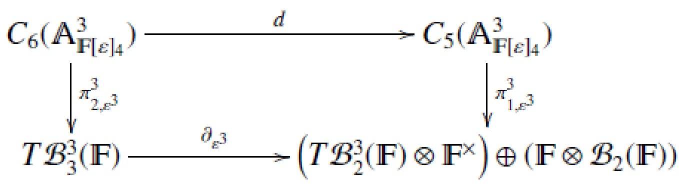
$$-3\frac{\Delta(v_4^*,v_2^*,v_3^*)_\varepsilon^3}{\Delta(v_4,v_2,v_3)}\otimes(\Delta(v_0,v_3,v_4)\wedge\Delta(v_0,v_2,v_3)- \Delta(v_0,v_3,v_4)\wedge\Delta(v_0,v_2,v_4)-\Delta(v_0,v_2,v_4)\wedge\Delta(v_0,v_2,v_3) +\Delta(v_1,v_3,v_4)\wedge\Delta(v_1,v_2,v_4)-\Delta(v_1,v_3,v_4)\wedge \Delta(v_1,v_2,v_3)+\Delta(v_1,v_2,v_4)\wedge\Delta(v_1,v_2,v_3))$$

This is the simplified value of $\delta^1\circ\pi_{1,\varepsilon^3}^3(\mathbb{V}_{04}^*)$ (29). Move to the other map

$$\delta^2\circ\pi_{1,\varepsilon^2}^3(\mathbb{V}_{04}^*)=\sum_{i=0}^{4}(-1)^i\{\left(\frac{3b_{3i}}{a_i}-\frac{3b_{1i}b_{2i}}{a_i^2}+\frac{b_{1i}^3}{a_i^3}\right) \wedge\left(\frac{3b_{3i}}{1-a_i}-\frac{3b_{1i}b_{2i}}{(1-a_i)^2}+\frac{b_{1i}^3}{(1-a_i)^3}\right)\wedge x_i\} \quad (31)$$

This can be simplified by first the expanding the sum for $i=0,\dots,4$ and then using $p\wedge(q+r)=p\wedge q+p\wedge r$. The sum of (31) and (22) gives the value of *LHS* of the theorem which is identical with the *RHS* of theorem.

Theorem 2.3 *The diagram below commutes.*



i.e. $\pi_{1,\varepsilon^3}^3\circ d=\pi_{2,\varepsilon^3}^3\circ\partial_{\varepsilon^3}$

Proof. Initially we have 720 terms due to the expansion of the map π_{2,ε^3}^3 according to (8). But many of them are identical due to the symmetry $\Delta(v_i,v_j,v_k)=\Delta(v_i,v_k,v_j)=\Delta(v_j,v_i,v_k)$ ..., so there will remain only

120 distinct terms. According to the Definition (8), we have

$$\pi^3_{2,\varepsilon^3}(\mathbb{V}^*_{05}) = \frac{2}{45}\mathrm{Alt}_6\langle r_3(\mathbb{V}_{05}); r_{3,\varepsilon}(\mathbb{V}^*_{05}), r_{3,\varepsilon^2}(\mathbb{V}^*_{05}), r_{3,\varepsilon^3}(\mathbb{V}^*_{05})]^3_3 \quad (32)$$

Now

$$\partial_{\varepsilon^2}\circ\pi^3_{2,\varepsilon^2}(\mathbb{V}^*_{05}) = \frac{2}{45}\mathrm{Alt}_6\{\langle r_3(\mathbb{V}_{05}); r_{3,\varepsilon}(\mathbb{V}^*_{05}), r_{3,\varepsilon^2}(\mathbb{V}^*_{05}), r_{3,\varepsilon^3}(\mathbb{V}^*_{05})]^3_2 \otimes r_3(\mathbb{V}_{05}) + (3\frac{r_{3,\varepsilon^3}(\mathbb{V}^*_{05})}{r_3(\mathbb{V}_{05})} - 3\frac{r_{3,\varepsilon}(\mathbb{V}^*_{05})}{r_3(\mathbb{V}_{05})}\frac{r_{3,\varepsilon^2}(\mathbb{V}^*_{05})}{r_3(\mathbb{V}_{05})} + \frac{r^3_{3,\varepsilon}(\mathbb{V}^*_{05})}{r^3_3(\mathbb{V}_{05})}) \otimes [r_3(\mathbb{V}_{05})]_2\} \quad (33)$$

With the help of the equations (9), (10), (11) and (12) we can write

$$3\frac{r_{3,\varepsilon^3}(\mathbb{V}^*_{05})}{r_3(\mathbb{V}_{05})} - 3\frac{r_{3,\varepsilon}(\mathbb{V}^*_{05})}{r_3(\mathbb{V}_{05})}\frac{r_{3,\varepsilon^2}(\mathbb{V}^*_{05})}{r_3(\mathbb{V}_{05})} + \frac{r^3_{3,\varepsilon}(\mathbb{V}^*_{05})}{r^3_3(\mathbb{V}_{05})}$$
$$= 3(\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^3}}{(v_0v_1v_3)} + \frac{(v_1^*v_2^*v_4^*)_{\varepsilon^3}}{(v_1v_2v_4)} + \frac{(v_2^*v_0^*v_5^*)_{\varepsilon^3}}{(v_2v_0v_5)} - \frac{(v_0^*v_1^*v_4^*)_{\varepsilon^3}}{(v_0v_1v_4)} - \frac{(v_1^*v_2^*v_5^*)_{\varepsilon^3}}{(v_1v_2v_5)} - \frac{(v_2^*v_0^*v_3^*)_{\varepsilon^3}}{(v_2v_0v_3)} - \frac{(v_0^*v_1^*v_3^*)_{\varepsilon}}{(v_0v_1v_3)}\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^2}}{(v_0v_1v_3)} - \frac{(v_1^*v_2^*v_4^*)_{\varepsilon}}{(v_1v_2v_4)}\frac{(v_1^*v_2^*v_4^*)_{\varepsilon^2}}{(v_1v_2v_4)} - \frac{(v_2^*v_0^*v_5^*)_{\varepsilon}}{(v_2v_0v_5)}\frac{(v_2^*v_0^*v_5^*)_{\varepsilon^2}}{(v_2v_0v_5)} + \frac{(v_0^*v_1^*v_4^*)_{\varepsilon}}{(v_0v_1v_4)}\frac{(v_0^*v_1^*v_4^*)_{\varepsilon^2}}{(v_0v_1v_4)} + \frac{(v_1^*v_2^*v_5^*)_{\varepsilon}}{(v_1v_2v_5)}\frac{(v_1^*v_2^*v_5^*)_{\varepsilon^2}}{(v_1v_2v_5)} + \frac{(v_2^*v_0^*v_3^*)_{\varepsilon}}{(v_2v_0v_3)}\frac{(v_2^*v_0^*v_3^*)_{\varepsilon^2}}{(v_2v_0v_3)}) + \frac{(v_0^*v_1^*v_4^*)^3_{\varepsilon}}{(v_0v_1v_4)^3} + \frac{(v_1^*v_2^*v_5^*)^3_{\varepsilon}}{(v_1v_2v_5)^3} + \frac{(v_2^*v_0^*v_3^*)^3_{\varepsilon}}{(v_2v_0v_3)^3} - \frac{(v_0^*v_1^*v_3^*)^3_{\varepsilon}}{(v_0v_1v_3)^3} - \frac{(v_1^*v_2^*v_4^*)^3_{\varepsilon}}{(v_1v_2v_4)^3} - \frac{(v_2^*v_0^*v_5^*)^3_{\varepsilon}}{(v_2v_0v_5)^3} \quad (34)$$

Using this value (32) becomes

$$= \frac{2}{45}\mathrm{Alt}_6\{\langle r_3(\mathbb{V}_{05}); r_{3,\varepsilon}(\mathbb{V}^*_{05}), r_{3,\varepsilon^2}(\mathbb{V}^*_{05}), r_{3,\varepsilon^3}(\mathbb{V}^*_{05})]^3_2 \otimes r_3(\mathbb{V}_{05}) + 3(\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^3}}{(v_0v_1v_3)} + \frac{(v_1^*v_2^*v_4^*)_{\varepsilon^3}}{(v_1v_2v_4)} + \frac{(v_2^*v_0^*v_5^*)_{\varepsilon^3}}{(v_2v_0v_5)} - \frac{(v_0^*v_1^*v_4^*)_{\varepsilon^3}}{(v_0v_1v_4)} - \frac{(v_1^*v_2^*v_5^*)_{\varepsilon^3}}{(v_1v_2v_5)} - \frac{(v_2^*v_0^*v_3^*)_{\varepsilon^3}}{(v_2v_0v_3)} - \frac{(v_0^*v_1^*v_3^*)_{\varepsilon}}{(v_0v_1v_3)}\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^2}}{(v_0v_1v_3)} - \frac{(v_1^*v_2^*v_4^*)_{\varepsilon}}{(v_1v_2v_4)}\frac{(v_1^*v_2^*v_4^*)_{\varepsilon^2}}{(v_1v_2v_4)} - \frac{(v_2^*v_0^*v_5^*)_{\varepsilon}}{(v_2v_0v_5)}\frac{(v_2^*v_0^*v_5^*)_{\varepsilon^2}}{(v_2v_0v_5)} + \frac{(v_0^*v_1^*v_4^*)_{\varepsilon}}{(v_0v_1v_4)}\frac{(v_0^*v_1^*v_4^*)_{\varepsilon^2}}{(v_0v_1v_4)} + \frac{(v_1^*v_2^*v_5^*)_{\varepsilon}}{(v_1v_2v_5)}\frac{(v_1^*v_2^*v_5^*)_{\varepsilon^2}}{(v_1v_2v_5)} + \frac{(v_2^*v_0^*v_3^*)_{\varepsilon}}{(v_2v_0v_3)}\frac{(v_2^*v_0^*v_3^*)_{\varepsilon^2}}{(v_2v_0v_3)}) + \frac{(v_0^*v_1^*v_4^*)^3_{\varepsilon}}{(v_0v_1v_4)^3} + \frac{(v_1^*v_2^*v_5^*)^3_{\varepsilon}}{(v_1v_2v_5)^3} + \frac{(v_2^*v_0^*v_3^*)^3_{\varepsilon}}{(v_2v_0v_3)^3} - \frac{(v_0^*v_1^*v_3^*)^3_{\varepsilon}}{(v_0v_1v_3)^3} - \frac{(v_1^*v_2^*v_4^*)^3_{\varepsilon}}{(v_1v_2v_4)^3} - \frac{(v_2^*v_0^*v_5^*)^3_{\varepsilon}}{(v_2v_0v_5)^3}) \otimes [r_3(\mathbb{V}_{05})]_2\} \quad (36)$$

Since we are using the same technique already used in the proof of Theorem.5.6 of [5], it seems unnecessary to write all steps. Going through the same procedure we can write (33) as

$$= \frac{1}{3}\mathrm{Alt}_6\{\langle r(v_0|v_1v_2v_3v_4); r_\varepsilon(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^2}(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^3}(v_0^*|v_1^*v_2^*v_3^*v_4^*)]^3_2 \otimes (v_0v_1v_3) + \left(3\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^3}}{(v_0v_1v_3)} - 3\frac{(v_0^*v_1^*v_3^*)_{\varepsilon}}{(v_0v_1v_3)}\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^2}}{(v_0v_1v_3)} + \frac{(v_0^*v_1^*v_3^*)^3_{\varepsilon}}{(v_0v_1v_3)^3}\right) \otimes [r(v_0|v_1v_2v_3v_4)]_2\} \quad (38)$$

Before going to the left hand side we rewrite the map π^3_{1,ε^3} as (see Theorem.4.3.3 of [12])

$$\pi^3_{1,\varepsilon^3}(\mathbb{V}^*_{05}) = \frac{1}{3}\mathrm{Alt}_5\{\langle r(v_0|v_1v_2v_3v_4); r_\varepsilon(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^2}(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^3}(v_0^*|v_1^*v_2^*v_3^*v_4^*)]^3_2 \otimes (v_0v_1v_2) + \left(3\frac{(v_0^*v_1^*v_2^*)_{\varepsilon^3}}{(v_0v_1v_2)} - 3\frac{(v_0^*v_1^*v_2^*)_{\varepsilon}}{(v_0v_1v_2)}\frac{(v_0^*v_1^*v_2^*)_{\varepsilon^2}}{(v_0v_1v_2)} + \frac{(v_0^*v_1^*v_2^*)^3_{\varepsilon}}{(v_0v_1v_2)^3}\right) \otimes [r(v_0|v_1v_2v_3v_4)]_2\}\} \quad (39)$$

Now the left hand side of the theorem is

$$\pi^3_{1,\varepsilon^3}\circ d(\mathbb{V}^*_{05}) = \pi^3_{1,\varepsilon^3}\left(\sum_{i=0}^{5}(-1)^i(v_0^*,\dots,\hat{v}_i,\dots,v_5^*)\right)$$

Applying the definition (35) and using the cycle $(v_0,\dots,v_5)$ after the expansion of Alt_5 we get (32)

$$\pi^3_{1,\varepsilon^3}\circ d(\mathbb{V}^*_{05}) = \frac{1}{3}\mathrm{Alt}_6\{\langle r(v_0|v_1v_2v_3v_4); r_\varepsilon(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^2}(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^3}(v_0^*|v_1^*v_2^*v_3^*v_4^*)]^3_2 \otimes (v_0v_1v_2) + \left(3\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^3}}{(v_0v_1v_3)} - 3\frac{(v_0^*v_1^*v_2^*)_{\varepsilon}}{(v_0v_1v_2)}\frac{(v_0^*v_1^*v_2^*)_{\varepsilon^2}}{(v_0v_1v_2)} + \frac{(v_0^*v_1^*v_2^*)^3_{\varepsilon}}{(v_0v_1v_2)^3}\right) \otimes [r(v_0|v_1v_2v_3v_4)]_2\}\}$$

Using the cycle (v_2v_3) we get

$$\pi^3_{1,\varepsilon^3}\circ d(\mathbb{V}^*_{05}) = -\frac{1}{3}\mathrm{Alt}_6\{\langle r(v_0|v_1v_2v_3v_4); r_\varepsilon(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^2}(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^3}(v_0^*|v_1^*v_2^*v_3^*v_4^*)]^3_2 \otimes (v_0v_1v_3) + \left(3\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^3}}{(v_0v_1v_3)} - 3\frac{(v_0^*v_1^*v_3^*)_{\varepsilon}}{(v_0v_1v_3)}\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^2}}{(v_0v_1v_3)} + \frac{(v_0^*v_1^*v_3^*)^3_{\varepsilon}}{(v_0v_1v_3)^3}\right) \otimes [r(v_0|v_1v_2v_3v_4)]_2\}\}$$

At last applying the two term relation we get the required expression with correct sign

$$\pi^3_{1,\varepsilon^3}\circ d(\mathbb{V}^*_{05}) = \frac{1}{3}\mathrm{Alt}_6\{\langle r(v_0|v_1v_2v_3v_4); r_\varepsilon(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^2}(v_0^*|v_1^*v_2^*v_3^*v_4^*), r_{\varepsilon^3}(\quad (v_0v_1v_3) + \left(3\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^3}}{(v_0v_1v_3)} - 3\frac{(v_0^*v_1^*v_3^*)_{\varepsilon}}{(v_0v_1v_3)}\frac{(v_0^*v_1^*v_3^*)_{\varepsilon^2}}{(v_0v_1v_3)} + \frac{(v_0^*v_1^*v_3^*)^3_{\varepsilon}}{(v_0v_1v_3)^3}\right) \otimes [r(v_0|v_1v_2v_3v_4)]_2\}\}$$

Corollary 2.4 *Both of the following chains are complexes*

(33)

1. $C_5(A^4_{F[\varepsilon]_4}) \xrightarrow{d'} C_4(A^3_{F[\varepsilon]_4}) \xrightarrow{\pi^3_{0,\varepsilon^3}} F\otimes F^\times \oplus \wedge^3 F$
2. $C_6(A^4_{F[\varepsilon]_4}) \xrightarrow{d'} C_5(A^3_{F[\varepsilon]_4}) \xrightarrow{\pi^3_{1,\varepsilon^3}} (T\mathcal{B}^3_2(F)\otimes F^\times) \oplus (F\otimes \mathcal{B}_2(F))$

Proof. Direct calculations lead us to have

$$\pi^3_{1,\varepsilon^3}\circ d' = \pi^3_{0,\varepsilon^3}\circ d' = 0$$

III. CONCLUSION

(34)

This work exhibits that the tangent group $TB^3_3(F)$ and its associated relations are valid for the order three. The above results motivate us to compute higher order tangent groups and their relative identities for weight for different weights.

IV. REFERENCES

[1] Cathelineau, J-L., *The tangent complex to the Bloch-Suslin complex*, Bull. Soc. Math. France **135** (2007) 565-59

[2] Elbaz-Vincent, Ph., and Gangl, H, *On Poly(ana)logs I*, Compositio Mathematica, **130**, 161-210 (2002).

[3] Cathelineau, J-L., *The tangent complex to the Bloch-Suslin complex*, Bull. Soc. Math. France **135** (2007) 565-59

[4] Elbaz-Vincent, Ph., and Gangl, H, *On Poly(ana)logs I*, Compositio Mathematica, **130**, 161-210 (2002).

[5] [Goncharov, A. B., *Geometry of Configurations, Polylogarithms and Motivic Cohomology*, Adv. Math., **114**(1995) 197-318.

[6] Hussain, S. and Siddiqui, R. Projected Five Term Relation in $T\mathcal{B}_2^2(F)$, International Journal of Algebra, Vol. 6, 2012, no. 28, 1353 – 1363.

[7] Hussain, S. and Siddiqui, R. Grassmannian Complex and Second Order Tangent Complex, Punjab University Journal of Mathematics, Vol. 48(2), 2016, 91 - 111.

[8] Hussain S. and Siddiqui R., Projective Configurations and Variant of Cathelineau Complex , Journal of Prime Research in Mathematics, Vol. 6, 2016, 24 - 34.

[9] Hussain S. and Siddiqui, R. Morphisms Between Grassmannian Complex and Higher Order Tangent Complex, Communications in Mathematics and Applications 2019, Vol. 10, No. 3, 509-518, doi: 10.26713/cma.v10i3.1220.

[10]Khalid M., Mariam Sultana, Javed Khan and Azhar Iqbal Geometry of Tangential and Configuration Chain Complexes for Higher Weights, Punjab University Journal of Mathematics, Vol. 52(2), 2020, 29 - 55.

[11]Khalid M., J. Khan and A. Iqbal Generalized geometry of Goncharov and configuration complexes, Turk J. Math. (2018), Vol. 42 : 1509-1527, doi:10.3906/mat-1702-25.

[12] Siddiqui, R. Infinitesimal and tangent to polylogarithmic complexes for higher weight, AIMS Mathematics 2019, Vol. 4(4): 1248-1257. doi: 10.3934/math.2019.4.1248.

[13] Siddiqui R., Geometry of configurations in tangent groups, AIMS Mathematics 2020, Vol. 5(1): 522-545. doi: 10.3934/math.2020035

[14] Siddiqui, R. Configuration Complexes and Tangential and Infinitesimal versions of Polylogarithmic Complexes, Doctoral thesis, Durham University. Available at Durham E-Theses Online: http://etheses.dur.ac.uk/586/ (2010)

[15] Hussain, S. M. A.A Mahbub, N. J. Nasu, H. Zheng Stabilized lowest equal order mixed finite element method for the Oseen viscoelastic fluid flow , Advances in Difference Equations 2018: 461.

[16] Hussain, S, A. Batool, Md. A.A Mahbub, N. J. Nasu, H. Zheng, SUPG Approximation for the Oseen viscoelastic fluid flow with stabilized lowest equal order mixed finite element method, Mathematics 2019, 7(2), 128.

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