A NEW ELEGANT APPROACH FOR SOLVING PANTOGRAPH DELAY DIFFERENTIAL EQUATIONS

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Abstract

The Pantograph Delay Differential Equation (PDDE), which incorporates a linear functional argument, is the subject of this study, which is a generalization of a functional differential equation. A new Novel Analytical Method (NAM) is used in this article to solve the Pantograph Delay Differential Equation. This approach uses simple calculus to perform long-term computations and is unrelated to any recurrence relation, therefore we must be cautious regarding its convergence. The resultant solutions are more physically realistic since they solve non-linear problems without needing linearization, discretization, or perturbation. Furthermore, iterations may be converged to Exact Solutions rather quickly, resulting in more accurate findings. Several illustrated examples are provided below to demonstrate the technique's efficacy and dependability, especially in non-linear scenarios.

Keywords: Pantograph Delay Differential Equations, Novel Analytical Method, Tylor Series, Convergence Analysis.

1. Introduction

Some differential equations involving time delay are called Delay Differential Equations (DDEs). At a particular time, the behavior of an unknown variable in a differential equation, relies on the behavior of this variable at earlier times, resulting in a time delay in the system. Many academics have investigated this differential equation family and sought numerical and approximate solutions. Furthermore, Pantograph Delay Differential Equations (PDDEs) have gained a lot of attention in the area of DDEs. PDDEs were initially proposed by Ockendon and Tayler [1]. Research related to some PDDEs applications can easily be seen in [1-4].

Many scholars have investigated the numerical solution of PDDEs since their introduction and numerous applications. We'd like to highlight a few of the techniques that have been offered. Boubaker Polynomials were utilized to solve PDDEs in [5]. The study in [6] provided a sequence of functions for generalized PDDEs based on the Variational Iteration Approach. Sedaghat et al. [7] provided a numerical technique for a pantograph equation based on Transferred Chebyshev Polynomials. The author in [8] also concentrated on the Chebyshev Polynomial approach for PDDEs. Cevik [9] proposed a compound strategy for solving PDDEs that combines the Perturbation Method with an iteration algorithm. In [10, 11], the solution of high-order PDDEs was examined by the method of Exponential Polynomials. The authors of [12] described Homotopy Perturbation Approach for DDEs. The reproducing kernel was used to solve a DDEs in [13]. Authors of [14,15] solved Pantograph Equations in general form using the Bernoulli Collocation Method.

Bessel Polynomials were used in [16] to get an approximated solution to PDDEs with Variable Coefficients. In [17] to solve Pantograph Equations, the Jacobi Rational Gauss Collocation Technique was presented. The Runge-Kutta Techniques for a family of Neutral Infinite DDEs with varying proportional delays were introduced in [18,19]. Bernstein Polynomials were used in [20,21] to approximation solve the Generalized Pantograph Equations. In addition, in [22], the Hermite Polynomials were introduced to provide numerical solutions to Pantograph Problem in a generalized form with Variable Coefficients. In [23, 24], writers explored the stability of ɵ-methods for solving a Generalized Pantograph Problem. In [8], the Tau Approach and the Chebyshev Polynomials were proposed for solving Pantograph Equations. For the Pantograph Equations, Xu and Huang [25, 26] discovered Discontinuous and Continuous Galerkin Solutions. Manuscript [3] explored the numerical solution of Pantograph Equations by using Trapezoidal Rule discretization. Rational Functions were used in [17,27] to estimate a Generalized Pantograph Equation over a semi-infinite interval. Furthermore, in studies mentioned in [28-30], Taylor Polynomials were utilized to approximate the solution of the Pantograph Equations.

Despite the strategies given above, a proficient and convergent method with low complexity and high accuracy is required to solve PDDEs. In this paper, we provide a Novel Analytical Approach, proposed by [31], for solving Pantograph-Type Delay Differential Equations numerically. Several academics labeled this novel approach as a Novel Analytical Method [32-34,36]. Recently, two of the paper's authors examined approximate solutions to Time Space Fractional Linear and Nonlinear KdV Equations [35]. The Taylor Series is used as an effective tool for solving nonlinear equations in this technique. The suggested technique yields Taylor Series solutions by combining the linear and nonlinear parts and then proceeds using just the calculus of many variables. In comparison to the Adomain Decomposition Method, this new variation is believed to be more straightforward to grasp. A thorough survey, on the other hand, indicates that the Delay Differential Equation has not yet been examined using this approach. These are the objectives for the current study, which focuses on solving PDDEs. This research includes a variety of numerical examples. It also compares the associated findings to the solutions (available in the existing literature) to prove the suggested technique's exceptional accuracy. The proposed approach may be used for all DDEs.

The remainder of this research report is organized as follows. Section 2 provides a comprehensive review of the Novel Analytical Method (NAM). Its convergence is also covered in section 3. The approach for solving a class of Pantograph Delay Differential Equations with graphical representation is shown in Section 4. Section 5 will conclude with some suggestions.

2. Description of Proposed Methodology

Consider Pantograph Delay Differential Equations of second order

$$\xi_{tt}(t) = \Psi(k\xi_t, kt, k\xi, \dots) \quad (1)$$

with initial conditions

$$\xi(0) = \xi_0 \text{ and } \xi_t(0) = \xi_1 \quad (2)$$

Integrating both sides of Eq. (1) from 0 to t, we get

$$\xi_t(t) = \xi_1 + \int_0^t \Psi[\xi]\, dt \quad (3)$$

Where $\Psi[\xi] = \Psi(k\xi_t, kt, k\xi, \dots)$

Again integrating both sides of Eq. (3) from 0 to t, we get

$$\xi(t) = \xi_0 + \xi_1 t + \int_0^t \int_0^t \Psi[\xi]\, dtdt \quad (4)$$

The Taylor series is extended for $\Psi[\xi]$ about $t = 0$ which is

$$\Psi[\xi] = \Psi[\xi_0] + \Psi'[\xi_0]t + \Psi''[\xi_0]\frac{t^2}{2!} + \Psi'''[\xi_0]\frac{t^3}{3!} + \cdots + \Psi^{(n)}[\xi_0]\frac{t^n}{n!} + \cdots \quad (5)$$

Now putting $\Psi[\xi]$ into Eq. (4), we get

$$\xi(t) = \xi_0 + \xi_1 t + \int_0^t \int_0^t \left[\Psi[\xi_0] + \Psi'[\xi_0]t + \Psi''[\xi_0]\frac{t^2}{2!} + \Psi'''[\xi_0]\frac{t^3}{3!} + \cdots + \Psi^{(n)}[\xi_0]\frac{t^n}{n!} + \cdots\right] dtdt$$

$$\xi(t) = \xi_0 + \xi_1 t + \Psi[\xi_0]\frac{t^2}{2!} + \Psi'[\xi_0]\frac{t^3}{3!} + \Psi''[\xi_0]\frac{t^4}{4!} + \Psi'''[\xi_0]\frac{t^5}{5!} + \cdots + \Psi^{(n-2)}[\xi_0]\frac{t^n}{n!} + \cdots$$

$$\xi(t) = a_0 + a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + a_4\frac{t^4}{4!} + a_5\frac{t^5}{5!} + \cdots + a_n\frac{t^n}{n!} + \cdots \quad (6)$$

Where

$$a_0 = \xi_0$$
$$a_1 = \xi_1$$
$$a_2 = \Psi[\xi_0]$$
$$a_3 = \Psi''[\xi_0]$$
$$\vdots$$
$$a_n = \Psi^{(n-2)}[\xi_0]$$

Where n denotes the highest derivative of $\xi(t)$. By the expansion of Eq. (6) with the help of Taylor's Series about $t = 0$ for ξ, we have

$$a_0 = \xi(0)$$
$$a_1 = \frac{d}{dt}\xi(0)$$
$$a_2 = \frac{d^2}{dt^2}\xi(0)$$
$$a_3 = \frac{d^3}{dt^3}\xi(0)$$
$$\vdots$$
$$a_n = \frac{d^n}{dt^n}\xi(0)$$

3. Convergence of Proposed Method

Consider the Pantagraph Delay Differential Equation in the following form

$$\xi(t) = \Psi\big(k\xi(t)\big), \quad (7)$$

where Ψ is a nonlinear operator. Then the solution obtained by the presented technique is equivalent to the following sequence $\omega_q = \sum_{p=0}^{q}\xi_p = \sum_{p=0}^{q}\delta_p\frac{(\Delta t)^p}{(p)!}$.

Theorem 3.1: Let H be a Hilbert space. Consider an operator $\Psi: H \to H$ such that it admits a solution in the form of ξ as mentioned in (7). The approximate solution $\sum_{p=0}^{\infty}\xi_p = \sum_{p=0}^{\infty}\delta_p\frac{(\Delta t)^p}{(p)!}$ is converged to the exact solution of ξ, when $\exists\, 0 \le \delta < 1, \|\xi_{p+1}\| \le \delta\|\xi_p\| \;\forall\, p \in \mathbb{N} \cup \{0\}$.

proof: we will prove that $\{S_q\}_{q=0}^{\infty}$ is a converged to Cauchy Sequence,

$$\|\omega_{q+1}-\omega_q\| = \|\xi_{q+1}\| \leq \delta\|\xi_q\| \leq \delta^2\|\xi_{q-1}\| \leq \cdots \leq \delta^q\|\xi_1\| \leq \delta^{q+1}\|\xi_0\|.$$

Now for $q, r \in \mathbb{N}, q \geq r$, we get

$$\begin{aligned}\|\omega_q-\omega_r\| &= \|(\omega_q-\omega_{q-1})+(\omega_{q-1}-\omega_{q-2})+\cdots+(\omega_{r+1}-\omega_r)\|\\ &\leq \|(\omega_q-\omega_{q-1})\|+\|(\omega_{q-1}-\omega_{q-2})\|+\cdots+\|(\omega_{r+1}-\omega_r)\|\\ &\leq \delta^q\|\xi_0\|+\delta^{q-1}\|\xi_0\|+\cdots+\delta^{r+1}\|\xi_0\| \leq (\delta^{r+1}+\delta^{r+2}+\cdots+\delta^q)\|\xi_0\|\\ &= \delta^{r+1}\frac{1-\delta^{q-r}}{1-\delta}\|\xi_0\|\end{aligned}$$

Hence $\lim_{q,r\to\infty}\|\omega_q-\omega_r\| = 0$ i.e., $\{\omega_q\}_{q=0}^{\infty}$ is a Cauchy Sequence in Hilbert Space H. Thus there exists $\omega \in H$ such that $\lim_{q\to\infty}\omega_q = \omega$ where $\omega = \xi$.

Definition 3.2: For every $q \in \mathbb{N} \cup \{0\}$, we define $\delta_q = \begin{cases}\frac{\|\xi_{q+1}\|}{\|\xi_q\|} & ,\|\xi_q\| \neq 0\\ 0, & otherwise\end{cases}$.

Corollary 3.3: From theorem 3.1, $\sum_{p=0}^{\infty}\xi_p = \sum_{p=0}^{\infty}\delta_n\frac{(\triangle t)^p}{(p)!}$ is converged to exact solution ξ when $0 \leq \delta_p < 1, p = 0,1,2,\ldots$. So, the analytical solution converges. In addition, it can be further verified by the computational results presented in the form of figures and graphs below.

4. Numerical Application

Example 4.1. Consider the following Nonlinear Pantagraph Delay Differential Equation of first order [37]:

$$\xi'(t) = A\xi(t) + B\xi(kt) + \cos(t) - A\sin(t) - B\sin(kt) \tag{8}$$

where $0 \leq t$ and $0 < k < 1$ with initial condition $\xi(0) = 0$. The exact solution of Eq. (8) is $\xi(t) = \sin(t)$. We assumed that $A = -1$ and $B = \frac{1}{2}$. Calculated the higher-order derivatives of Eq. (8) then, put $t = 0$ in each derivatives term, and by using initial conditions, we obtained the following results:

$$\begin{aligned}a_0 &= 0\\ a_1 &= 1\\ a_2 &= 0\\ a_3 &= -1\\ a_4 &= 0\\ &\vdots\end{aligned} \tag{9}$$

Putting Eq. (9) into Eq. (6), we get

$$\xi'(t) = 1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \cdots \quad (10)$$

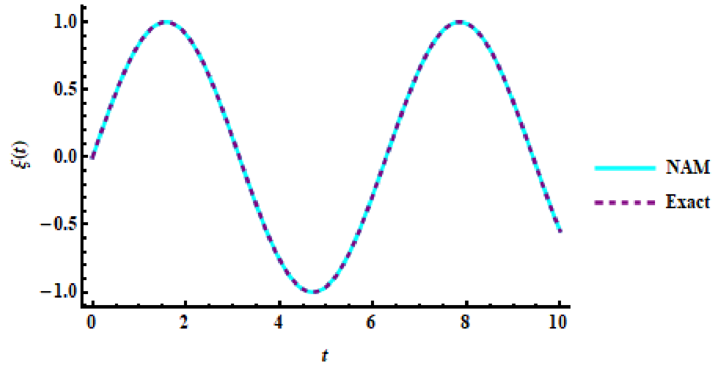

Figure 1. Graphical comparison of the Exact solution and numerical solution for Example 4.1.

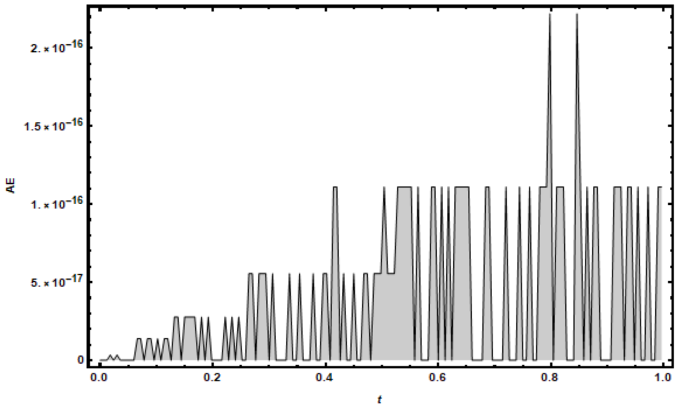

Figure 2. Absolute Error graph of the obtained numerical results by using NAM for Example 4.1.

From Figure 1, we can see that NAM is very closely equal to the exact solution of test Example 4.1. It has been also observed that the numerical solution converges to the exact solution as the number of iterations increases at $k = \frac{1}{2}$. Figure 2 shows the absolute errors of Example 4.1 at t ranges from zero to unity. It is seen from figure 2 that we have very small absolute errors in $t \epsilon [0,1]$ by using the Novel Analytical Method, which shows its shows accuracy in convergence. Table 1 shows the agreement between the obtained approximate solutions and the exact ones via calculating the $absolute\ errors = \left|\xi_{exact}(t) - \xi_{approx}(t)\right|$.

Table 1. Comparison between exact and NAM for Example 4.1.

t	Exact	NAM	Absolute Error
0.2	0.198669	0.198669	2.77556×10^{-17}
0.4	0.389418	0.389418	5.55112×10^{-17}
0.6	0.564642	0.564642	0.00
0.8	0.717356	0.717356	1.11022×10^{-16}
1.0	0.841471	0.841471	0.00

Example 4.2. Consider the following Nonlinear Pantograph Delay Differential Equation of first order [37]:

$$\xi'(t) = A\xi(t) + Be^{kt}\,\xi(kt) \tag{11}$$

where $0 \le t$ and $0 < k < 1$ with initial condition $\xi(0) = 1$. The exact solution of Eq. (11) is $\xi(t) = e^t$ at $k = \frac{1}{2}$. We assumed that $A = \frac{1}{2}$ and $B = \frac{1}{2}$. Calculated the higher-order derivatives

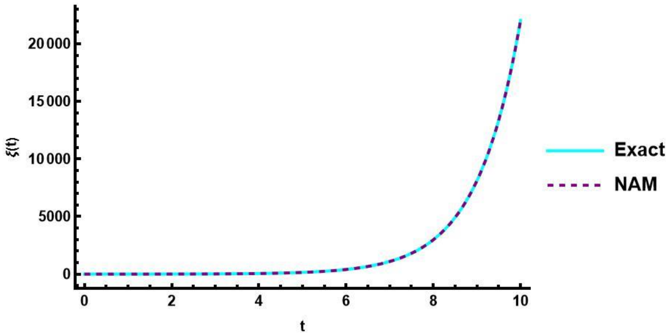

Figure 3. Graphical comparison of the Exact solution and numerical solution for Example 4.2. by using NAM.

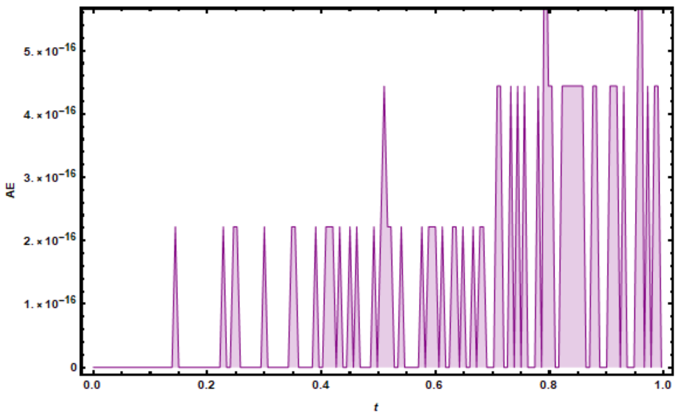

Figure 4. Absolute Error graph of the obtained numerical results by using NAM for Example 4.2.

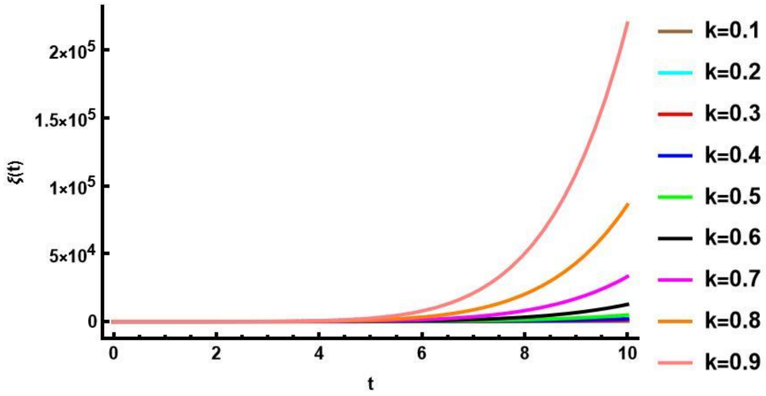

Figure 5. Different values of k were obtained numerically by using NAM for Example 4.2.

of Eq. (11) then, put $t = 0$ in each derivatives term and by using initial conditions, we obtained the following results:

$$\begin{aligned} a_0 &= 1 \\ a_2 &= \frac{1}{2} + k \\ a_3 &= \frac{3k^2}{2} + \frac{1}{2}(\frac{1}{2} + k) + \frac{1}{2}k^2(\frac{1}{2} + k) \\ &\vdots \end{aligned} \tag{12}$$

Putting Eq. (12) into Eq. (6), we get

$$\xi(t) = 1 + t + \frac{1}{4}(1 + 2\,k)t^2 + \frac{1}{24}(1 + 2\,k + 7\,k^2 + 2\,k^3)t^3 + \frac{1}{192}(1 + 2\,k + 7\,k^2 + 25\,k^3 + 14\,k^4 + 7\,k^5 + 2\,k^6)t^4 + \left(\frac{1}{1920}\right)(1 + 2\,k + 7\,k^2 + 25\,k^3 + 87\,k^4 + 73\,k^5 + 65\,k^6 + 41\,k^7 + 14\,k^8 + 7\,k^9 + 2\,k^{10})t^5 + \cdots \tag{13}$$

Table 2. Comparison between exact and NAM for Example 4.2.

t	**Exact**	**NAM**	**Absolute Error**
0.2	1.2214	1.2214	0.00
0.4	1.49182	1.49182	0.00
0.6	1.82212	1.82212	2.22045×10^{-16}
0.8	2.22554	2.22554	4.44089×10^{-16}
1.0	2.71828	2.71828	4.44089×10^{-16}

At $k = \frac{1}{2}$, Eq. (13) will be exactly the same as the exact solution of Example 4.2. The graphical representation of numerical simulations of the behavior of the exact and approximate solution (obtained by NAM) is shown in Figure 3. Absolute errors are plotted in Figure 4 in to understand the convergence of the method. In Figure 5, we plot the graphs of different values of k with $A = \frac{1}{2}$ and $B = \frac{1}{2}$ It has been observed that graphs behave approximately the same for all values of k till $t = 5$ but graph slopes upward as k increases for values of $t > 5$. Table 2 illustrates the agreement between the obtained approximate solutions and the exact ones via evaluating the $absolute\ errors = |\xi_{exact}(t) - \xi_{approx}(t)|$.

Example 4.3. Consider the following Nonlinear Pantograph Delay Differential Equation of first order [37]:

$$\xi'(t) = A\xi(kt) - \xi(t) + Be^{-kt} \quad (14)$$

where $0 \le t$ and $0 < k < 1$ with initial condition $\xi(0) = 1$. The exact solution of Eq. (14) is

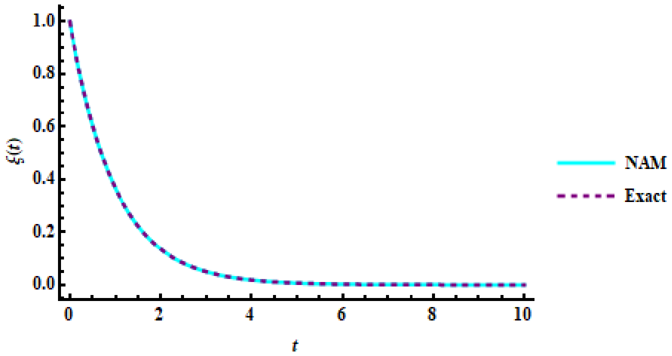

Figure 6. Graphical illustration of comparison of the exact solution and obtained numerical solution for Example 4.3.

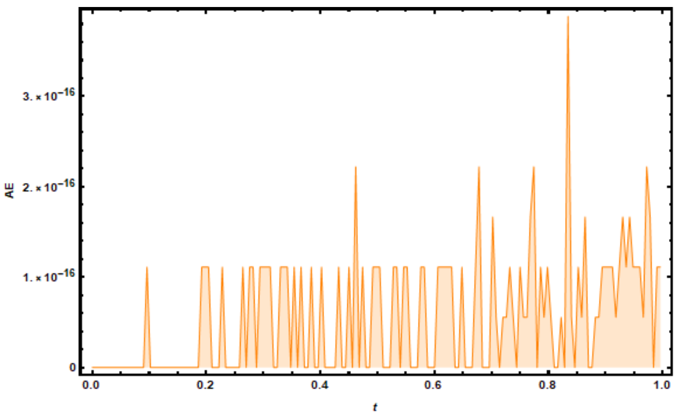

Figure 7. Absolute error graph of the obtained numerical results for Example 4.3.by novel analytical method.

$\xi(t) = e^{-t}$. We assumed that $A = \frac{1}{2}$, $B = -\frac{1}{2}$ and $k = \frac{1}{2}$. Calculated the higher-order derivatives of Eq. (14) then, put $t = 0$ in each derivatives term, and by using initial conditions, we obtained the following results:

$$a_0 = 1 \quad (15)$$

$$a_1 = -1$$
$$a_2 = 1$$
$$a_3 = -1$$
$$\vdots$$

Putting Eq. (15) into Eq. (6), we get

$$\xi(t) = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + (-1)^n \frac{t^n}{n!} + \cdots \quad (16)$$

Table 3. Comparison between the exact solution and solution by NAM for Example 4.3.

t	Exact	NAM	Absolute Error
0.2	0.818731	0.818731	1.11022×10^{-16}
0.4	0.67032	0.67032	0.
0.6	0.548812	0.548812	1.11022×10^{-16}
0.8	0.449329	0.449329	1.11022×10^{-16}
1.0	0.367879	0.367879	1.11022×10^{-16}

In figure 6, at $k = \frac{1}{2}$, the comparison of the exact and nnumerical solutions obtained by NAM describes the accuracy of the method. Absolute errors in Figure 7 show that the approximate solution is entirely in harmony with the exact solution of test Example 4.3. Table 3 shows the absolute errors that are obtained for some selected various values of t for Example 4.3

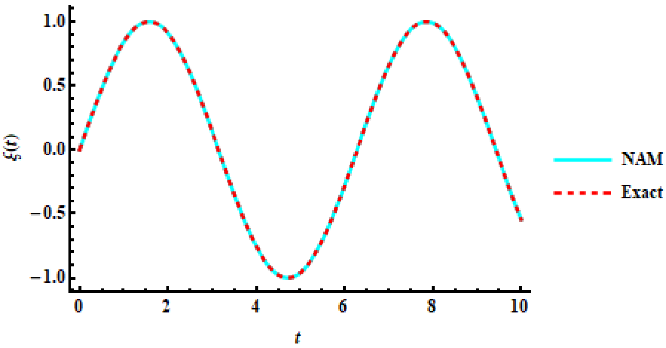

Figure 8. Comparison of the exact solution and obtained numerical solution for Example 4.4.

Example 4.4. Consider the following Nonlinear Pantograph Delay Differential Equation of third order [38]:

$$\xi'''(t) = -1 + 2\xi^2(kt) \quad (17)$$

where $0 \leq t$ and $0 < k < 1$ with initial conditions $\xi(0) = 0$, $\xi'(0) = 1$ and $\xi''(0) = 0$. The exact solution of Eq. (17) is $\xi(t) = \sin(t)$ at $k = \frac{1}{2}$. Calculated the higher-order derivatives of

Eq. (17) then, put $t = 0$ in each derivatives term, and by using initial conditions, we obtained the following results:

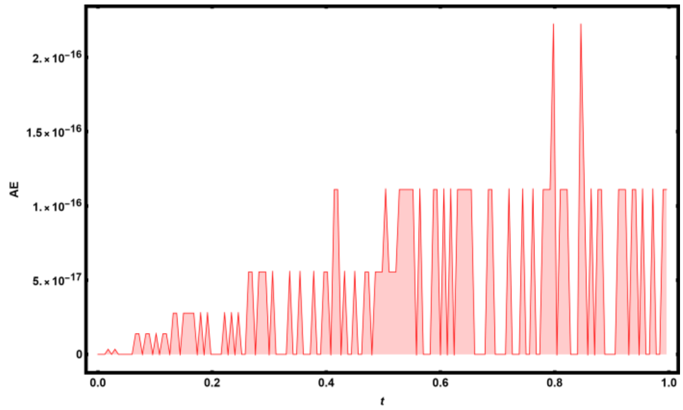

Figure 9. Absolute error graph of the obtained numerical results for Example 4.4. by novel analytical method.

$$\begin{gathered} a_0 = 0 \\ a_1 = 1 \\ a_2 = 0 \\ a_3 = -1 \\ a_4 = 0 \\ a_5 = 4k^2 \\ a_6 = 0 \\ a_7 = -16k^4 \\ \vdots \end{gathered} \tag{18}$$

Putting Eq. (18) into Eq. (6), we get

$$\xi(t) = t - \frac{t^3}{6} + \frac{k^2 t^5}{30} - \frac{k^4 t^7}{315} + \frac{k^6(5 + 12\,k^2)t^9}{45360} + \cdots \tag{19}$$

In figure 8, the exact and the approximate solution coincide with each other. Figure 9 evidence that NAM converges to an exact solution by showing the lowest values of Absolute errors. Graphical illustration of numerical solution of test Example 4.4 is given in Figure 10 for different values of k. Deviation can be observed for the values of $t > 6$ in this plot for higher values of k. Table 4 demonstrates the agreement between the obtained approximate solutions by NAM and the exact by calculating the $absolute\ errors = \left|\xi_{exact}(t) - \xi_{approx}(t)\right|$.

Table 4. Comparison between exact and NAM for Example 4.4.

t	Exact	NAM	Absolute Error
0.2	0.198669	0.198669	2.77556×10^{-17}
0.4	0.389418	0.389418	5.55112×10^{-17}
0.6	0.564642	0.564642	0.00
0.8	0.717356	0.717356	1.11022×10^{-16}
1.0	0.841471	0.841471	0.00

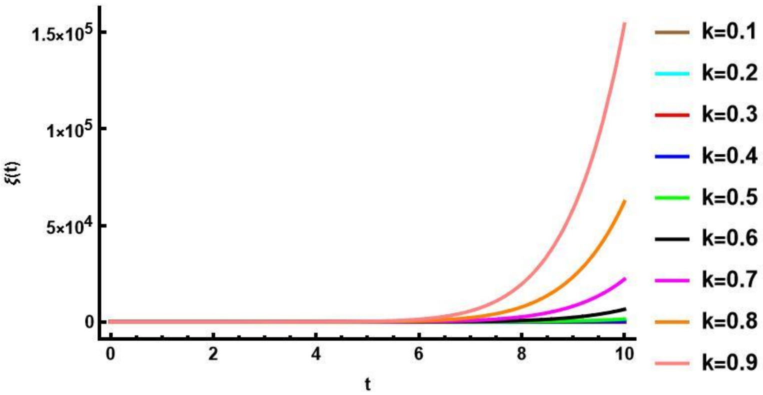

Figure 10. Different values of k were obtained numerically by using NAM for Example 4.4.

Example 4.5. Consider the following Multi-Pantograph Delay Differential Equation of the first order[38]:

$$\xi'(t) = -\frac{5}{6}\xi(t) + 4\,\xi(k_1 t) + 9\xi(k_2 t) + t^2 - 1 \tag{20}$$

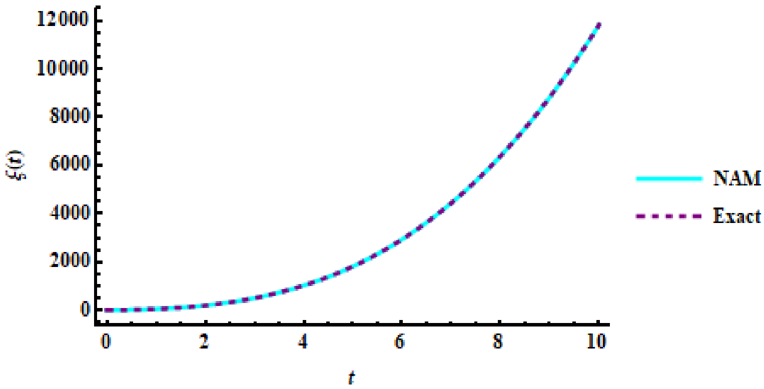

Figure 11. Graph of the exact solution versus obtained numerical solution for Example 4.5. by using NAM.

where $0 \le t$ and $0 < k < 1$ with initial condition $\xi(0) = 1$. The exact solution of Eq. (20) is $\xi(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3$. We assumed that $k_1 = \frac{1}{2}$ and $k_2 = \frac{1}{3}$. Calculated the higher-order derivatives of Eq. (20) then, put $t = 0$ in each derivatives term, and by using the initial condition, we obtained the following results:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \frac{67}{6} \\ a_2 &= -\frac{335}{36} + \frac{134k_1}{3} + \frac{201k_2}{2} \\ a_3 &= 2 - \frac{335}{216}(-5 + 24k_1 + 54k_2) + \frac{67}{9}k_1^2(-5 + 24k_1 + 54k_2) + \frac{67}{4}k_2^2(-5 + 24k_1 \\ &\quad + 54k_2) \\ a_4 &= 0 \\ a_5 &= 0 \\ &\vdots \end{aligned} \tag{21}$$

Putting Eq. (21) into Eq. (6), we get

$$\begin{aligned} \xi(t) = {} & \frac{24715}{7776} + \frac{67t}{6} + \frac{871t^2}{72} + \frac{1303t^3}{1296} - \frac{1303t^4}{10368} + \frac{871k_1}{324} + \frac{67t^2k_1}{3} + \frac{67t^3k_1}{54} \\ & - \frac{67t^4k_1}{432} + \frac{11323k_1^2}{324} + \frac{871}{54}t^3k_1^2 - \frac{871}{432}t^4k_1^2 + \frac{45773k_1^3}{972} + \frac{268}{9}t^3k_1^3 \\ & - \frac{3521t^4k_1^3}{1296} - \frac{10877k_1^4}{324} + \frac{67}{54}t^4k_1^4 - \frac{23852k_1^5}{81} + \frac{871}{54}t^4k_1^5 - \frac{19162k_1^6}{27} \\ & + \frac{268}{9}t^4k_1^6 - \frac{7042k_1^7}{27} + \frac{1072k_1^8}{9} + \frac{13936k_1^9}{9} + \frac{8576k_1^{10}}{3} \end{aligned} \tag{22}$$

which is the same as an exact solution.

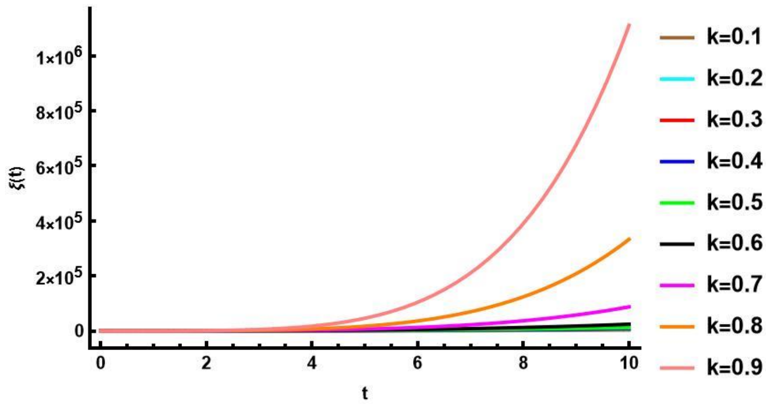

Figure 12. Different values of k, are obtained numerically by using NAM for Example 4.5.

When we fix $k_2 = \frac{1}{3}$ and $k_1 = \frac{1}{2}$ in an approximate solution of Example 4.5, then it will be precise as the same as the exact solution, which can be seen in Figure 11. In Figure 12, we plot the graph of different values of k_1 by fixing $k_2 = \frac{1}{3}$, we may observe that as k approaches zero then slopes downwards, but as k increases, the graph slopes upward.

Example 4.6. Consider the following Pantograph Delay Differential Equation of the Second order[38]:

$$\xi''(t) = \frac{3}{4}\xi(t) + \xi(kt) - \frac{\frac{3}{4}}{t^2 + t + 1} - \frac{4}{t^2 + 2t + 4} + \frac{2(2t+1)^2}{(t^2+t+1)^3} - \frac{2}{(t^2+t+1)^2} \quad (23)$$

where $0 \leq t$ and $0 < k < 1$ with initial conditions $\xi(0) = 1$ and $\xi'(0) = -1$.

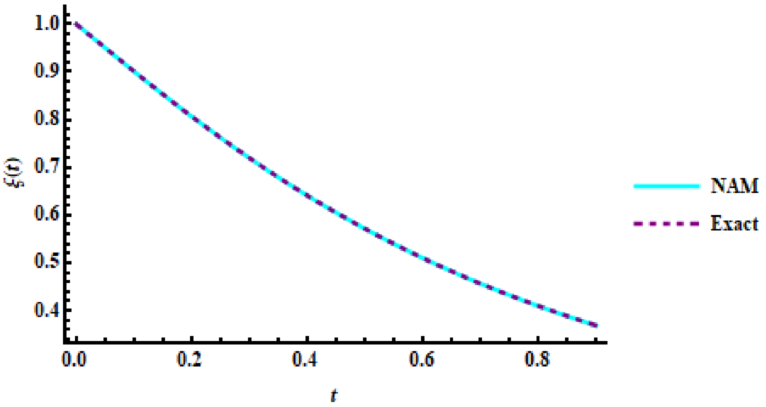

Figure 13. Numerical comparison of the exact solution and obtained numerical solution by using NAM for Example 4.6.

The exact solution of Eq. (23) is $\xi(t) = \frac{1}{t^2+t+1}$ at $k = \frac{1}{2}$. We assumed that $k = \frac{1}{2}$. Calculated the higher-order derivatives of Eq. (23) then, put $t = 0$ in each derivatives term, and by using the initial conditions, we obtained the following results:

$$\begin{gathered} a_0 = 1 \\ a_1 = -1 \\ a_2 = 0 \\ a_3 = \frac{13}{2} - k \\ a_4 = -24 \\ a_5 = \frac{1}{8}(-3 - 6k + 52k^3 - 8k^4) \\ \vdots \end{gathered} \quad (24)$$

Putting Eq. (24) into Eq. (6), we get

$$\xi(t) = \xi(t) = 1 - t - \frac{1}{12}(-13 + 2\,k)t^3 - t^4 - \frac{1}{960}(3 + 6\,k - 52\,k^3 + 8\,k^4)t^5 - \frac{1}{480}(-481 + 16\,k^4)t^6 + \cdots \quad (25)$$

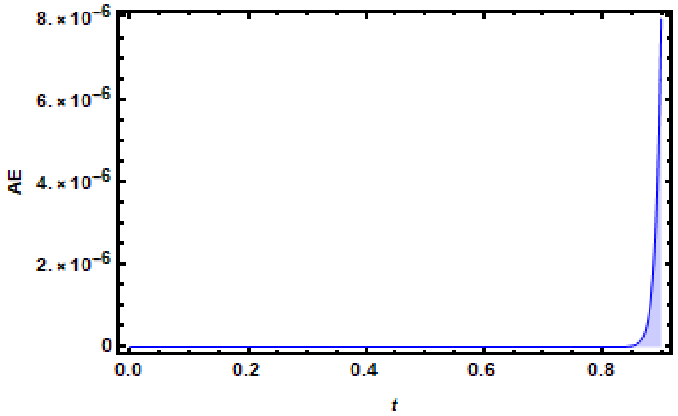

Figure 14. Absolute Error graph of the obtained numerical results by using NAM for Example 4.6.

Table 5. Comparison between exact and NAM for Example 4.6.

t	Exact	NAM	Absolute Error
0.2	0.806452	0.806452	1.11022×10^{-16}
0.4	0.641026	0.641026	0.00
0.6	0.510204	0.510204	1.11022×10^{-16}
0.8	0.409836	0.409836	1.11022×10^{-16}

which is the same as the exact solution. Figure 13 shows that the new analytical method gives accurate results when applied on test problem 4.6 with $k = \frac{1}{2}$ as both graphs of the exact solution and numerical solution coincide with each other. Absolute errors in figure 14 show the accuracy of the method. Table 5 demonstrates the harmony between the obtained approximate solutions by NAM and the exact ones by computing the $absolute\ errors = |\xi_{exact}(t) - \xi_{approx}(t)|$.

5. Conclusions

In this paper, we introduced a novel analytical approach to pantograph delay differential equations. The practicality and convergence of estimated solutions found were explored. Some pantograph delay differential equations have been successfully solved Novel Analytical

Method (NAM). When the resulting findings were compared to exact solutions, we discovered that our method is more exact than certain current methodologies. The suggested approach more directly computes the Taylor series' coefficients. Other methods may require additional calculations and complications to give a more precise result. Thus, in a few terms, the suggested technique outperforms the other methods in terms of accuracy. The resultant solutions are more physically realistic since it solves non-linear problems without discretization, linearization, or perturbation. In the future, we want to apply the approach to the numerical solution of additional forms of delay differential equations, such as fractional partial delay differential equations and fractional delay integral differential equations

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