Mathematical Modelling of the Lifetime Data Sets with Probability distributions

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Abstract- In practice there does exist some data sets with extreme values like in environmental sciences, and engineering etc. If we deal with such type of datasets with the classical methods, then the results may not be more efficient as compared to the specialized methods. This paper concentrated on the analysis of the real data sets with extreme and without extreme values by using probability distributions. It has been determined that the Flexible Exponential Type Exponential (FETE) distribution performs well as compared to others in both cases. Various statistical properties (r^{th} moments, survival function, hazard function, etc) and a simulation study is conducted on the efficient model. Furthermore, the proposed model is proficient to display the non-monotonic hazard rate with the lifetime data. The parameters are estimated using maximum likelihood method.

Index Terms- Exponential distribution, FE, Modelling, Statistical Properties.

I. Introduction

It is a usual practice to deals with different kinds of probability distributions so as to predict and determine the future behaviors of data sets. These models have vast applications in the field of biological sciences, actuarial sciences, and engineering among many other fields. Among these distributions, one of the prominent and simple distribution is the exponential distribution. The exponential distribution has a few attractive properties, such as the lack of memory property. However, one of the major disadvantages of the exponential distribution is that it is unable to model no monotonic data due to its constant hazard rate function. In many practical circumstances, non-monotone hazard functions may be observed, and in such cases, the exponential distribution cannot be applied. Due to this limitation, many researchers have introduced the modified form of the exponential distribution. For example, [1] studied the beta exponential distribution, [2] proposed the exponentiated exponential distribution, [3] investigated the Kumaraswamy exponential distribution, [4] proposed the weighted exponential distribution, [5] combined the logarithmic distribution with an exponential distribution and developed the exponential logarithmic distribution, [6] discussed beta generalized exponential distribution. [7] proposed the gamma exponential distribution, Weibull exponential (WE) distribution introduced by [8], Marshall–Olkin generalized exponential distribution which were defined by [9], Kumaraswamy Marshall–Olkin exponential distribution which were given by [10].

The exponential distribution is mainly concerned with estimating the time duration when an event may occur. It is prominently used by earth scientists to calculate the approximate time when an earthquake is likely to occur in a specific area. The exponential distribution is most used to determine the reliability of electronic devices such as a laptop, processor, mobile phone, and so on. It assists the engineers in calculating an approximate time after which the product will rupture. Similarly, assume that a new customer enters a store every two minutes on average. Determine the likelihood that a new customer will arrive in less than one minute after a customer arrives, we will use the exponential distribution.

As examples, some researchers have suggested techniques for adding probability models. This phenomenon of parameter addition creates more robust families of distributions, which are efficiently employed for modeling data sets in engineering, economics, biological research, and environmental sciences. Therefore, in this regard, some famous classes are the Kumaraswamy class of distributions proposed by [11], odd Fréchet class of distributions studied by [12], odd generalized N-H class of distributions investigated by [13], T -X class of distributions introduced by [14], transmuted odd Fréchet class of distributions proposed in [15], the Weibull class of distributions investigated in [16], exponentiated generalized alpha power class of distributions studied in [17], truncated Cauchy power Weibull class of distributions proposed in [18],odd Perks class of distributions studied in [19], exponentiated version of the M family of distributions investigated by [20], generalized exponential class of distributions proposed in [21], alpha power transformation class of distributions discussed in [22] among others.

The aim of this paper to choose a best model among lifetime distributions for modeling the real data sets with extreme values as well as clean information. Clean information adopts monotonic hazard rate functions whereas the data set with extreme values adopt a non-monotonic type hazard rate function.

Let X is continuous random variable having the cumulative distribution function (CDF). Then, the CDF of X can be defined as

$$F_X(x)=\begin{cases} e^{\alpha\left(1-\frac{1}{F(x)}\right)}; if\ \alpha>1 \\ e^{\left(1-\frac{1}{F(x)}\right)}; if\ \alpha=1 \end{cases} \quad (1)$$

Where, α is the scale parameter and $F(x)$ is the CDF of the baseline distribution. As the Exponential distribution can model only a constant failure rate but unable to model non-monotonic hazard rate function. To overcome these limitations, some modification in traditional exponential distribution has been done by incorporating the CDF of the exponential distribution in (1) and get the new CDF in the form of (3). The CDF of the exponential distribution is given as

$$F_E(x)=1-e^{-\beta x},\ x>0 \quad (2)$$

where, β is the scale parameter also called rate parameter.

II. FLEXIBLE EXPONENTIAL TYPE EXPONENTIAL (FETE) DISTRIBUTION

In this section, the special case of the CDF and PDF are derived by using Eq (2). Let a positive random variable x has the FETE distribution, then the CDF of the FETE distribution is formed with parameters α and β.

$$F(x)=e^{\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right)}, \qquad \alpha,\beta>0, x>0 \quad (3)$$

The corresponding PDF to (3) is given by,

$$f(x)=\frac{\alpha\beta e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{-\beta x}-1\right)^2}, \alpha,\beta>0, x>0 \quad (4)$$

The plots of the PDF and CDF of the FETE distribution, with various values of α and β are shown in Fig 1. From Fig. 1 the PDF can be unimodal and right skewness.

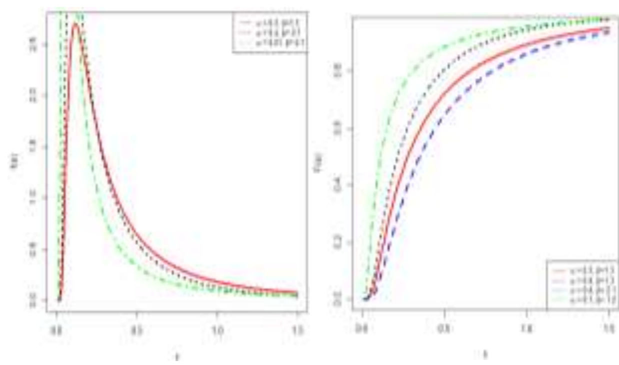

Fig 1: PDF & CDF of FETE Distribution

III. STATISTICAL PROPERTIES

This section presents few mathematical properties of the FETE distribution specially, quantile function, moments, rth moment, MRL function, Shannon Entropy, Survival and Failure function. The detailed derivations of these properties are given below.

R^th^ Moments- The rth moment μ_r of a random variable x follows the FETE distribution is given as

$$\mu_r=\sum_{k=1}^{\infty}\frac{(-1)^{kr}}{(k\beta)^r}\alpha^{kr}(-1)^{2kr}\overline{\big)(1-kr,-z)}\Big|_0^{\infty} \quad (5)$$

Proof.

The rth moment can be defined as.

$$\mu_r=\int_0^\infty x^r f(x)dx \text{ where, } f(x) \text{ is the pdf of FETE.}$$

$$\mu_r=\int_0^\infty x^r\frac{\alpha\beta e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{-\beta x}-1\right)^2}dx \quad (6)$$

$let\ z=-\frac{\alpha}{e^{\beta x}-1}$, then $dz=\frac{\alpha\beta e^{\beta x}}{\left(e^{\beta x}-1\right)^2}dx$, where

$$x=\frac{\log\left(\frac{z-\alpha}{z}\right)}{\beta}$$

Putting the above expressions in (6), we get

$$=\frac{1}{\beta^r}\int_\infty^0\left[\log\left(\frac{z-\alpha}{z}\right)\right]^r e^z dz \quad (7)$$

$$\left[\log\left(\frac{z-\alpha}{z}\right)\right]^r=\left(-\sum_{k=1}^{\infty}\frac{(-1)^k\left(-\alpha/z\right)^k}{k}\right)^r, \qquad for\left|\frac{\alpha}{z}\right|>1$$

by substituting this series representation in (7), we have

$$=\sum_{k=1}^{\infty}\frac{(-1)^{kr}}{(k\beta)^r}\int_0^\infty\left(-\alpha/z\right)^{kr}e^z dz$$

The final form of the FETW distribution is given below.

$$\mu_r = \sum_{k=1}^{\infty} \frac{(-1)^{kr}}{(k\beta)^r} \alpha^{kr} (-1)^{2kr} \overline{\big)} (1-kr,-z) \Big|_0^{\infty}$$

It was assuming that $1 \neq kr \ \& \ 2 \neq kr$

Quantile Function and Median- To generate the random numbers from the FETE distribution we have derived the quantile function. The quantile function can be defined as, $F(x) = u$ where, u (0, 1) is a uniform random number and $F(x)$ is the CDF of FETE which can be solved for random variable x.

$$P[X \le x] = u$$

$$e^{\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right)} = u \tag{8}$$

Taking log on both sides of (8), we have

$$\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right) = \log(u)$$

$$1-\frac{\alpha}{\alpha-\log(u)} = e^{-\beta x} \tag{9}$$

Again, taking log on both sides of (9), we get the final result as

$$x = \frac{-\log\left(1-\frac{\alpha}{\alpha-\log(u)}\right)}{\beta} \tag{10}$$

Equation (10) can be used to generate random numbers when the parameters $\alpha \& \beta$ are known. For Median, put $u = 1/2$ in equation (10) we have.

$$x = \frac{-\log\left(1-\frac{\alpha}{\alpha-\log(1/2)}\right)}{\beta}$$

Mode of the FETE Distribution- The mode of $FETE(\alpha,\beta)$ can be derived from equation (3) by taking derivative and equate them to zero. The mode can be defined as

$$\frac{d}{dx} f(x) = 0$$

$$\frac{d}{dx}\left(\frac{\alpha\beta e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{-\beta x}-1\right)^2}\right) = 0 \tag{11}$$

Solving equation (11) for x, we have

$$\frac{-\alpha\beta^2\left(e^{2\beta x}-\alpha e^{\beta x}-1\right)e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^4} = 0$$

$$x = \frac{\log\left(1/2\left(\sqrt{\alpha^2+4}+\alpha\right)\right)}{\beta}, \quad \beta > 0, x > 0$$

For the further description shape of the FETE distribution, the value of the skewness and kurtosis with various parameters values are listed below the table. From Table 1 the skewness and kurtosis are positive the distribution has right skewness.

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Table 1: Skewness & Kurtosis

Parameters	Skewness	Kurtosis
α =0.9, β =0.6	0.276538	1.381864
α =0.10, β =0.7	0.430512	1.82
α =0.12, β =0.17	0.422834	1.781082
α =1.12, β =1.17	0.257977	1.357318
α =1.50, β =1.70	0.234533	1.331488
α =3.3, β =3.4	0.183261	1.293043
α =4.3, β =4.4	0.17052	1.287117

Survival and Hazard Functions- The Survival or reliability function which is denoted by $S(x)$ is the probability that an item will not fail before a certain time t is defined by $S(x) = 1 - \mathrm{F}(x)$. The survival function of the FETE distribution is given by.

$$S(x) = 1 - F(x) \tag{12}$$

Substituting equation (3) in (12), we obtained the Survival function of the FETE distribution as

$$S(x) = 1 - e^{\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right)} \tag{13}$$

and the hazard rate (HF) function which is a ratio of the PDF to Survival function is defined as

$$h(x)=\frac{f(x)}{S(x)} \tag{14}$$

By putting (4) and (13) in (14) we have

$$H(x)=\frac{\dfrac{\alpha\beta e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{-\beta x}-1\right)^{2}}}{1-e^{\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right)}}.$$

The HF of the FETE distribution is given as

$$H(x)=\frac{\alpha\beta e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{-\beta x}-1\right)^{2}1-e^{\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right)}} \tag{15}$$

Fig.2 shows the graphical representation of the HF of the FETE distribution with distinct values of α and β . Fig. 2 show the HF can be up-side-down.

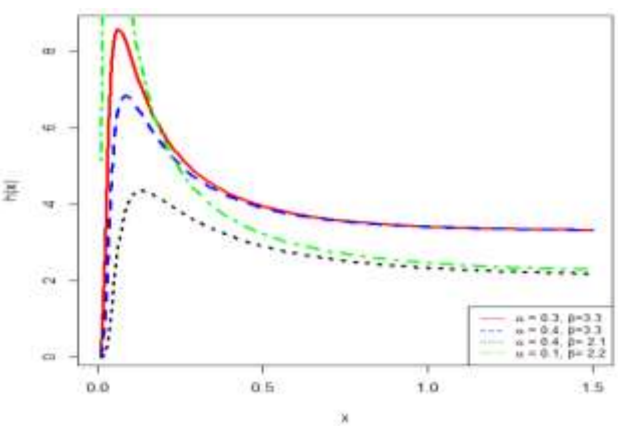

Fig 2: Hazard Function of FETE

Order Statistics- Let $X_1, X_2, \ldots, X_i, \ldots, X_n$ be ordered random variables, then the PDF of the i[th] order statistics is given as

$$f_{(i,n)}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \tag{16}$$

Using (3) and (4) in the above expression, we have

$$f_{(i,n)}(x)=\frac{n!}{(i-1)!(n-i)!}\left(\frac{\alpha\beta e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^{2}}\right)\left(e^{\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right)}\right)^{i-1}\left(1-e^{\alpha\left(1-\frac{1}{1-e^{-\beta x}}\right)}\right)^{n-i} \tag{17}$$

the smallest order PDF of the FETE distribution is

$$f_{(1,n)}(x)=\frac{\alpha\beta n\left(1-e^{-\frac{\alpha}{e^{\beta x}-1}}\right)^{n-1}e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^{2}} \tag{18}$$

and the higher-order PDF of the FETE distribution give as

$$f_{(n,n)}(x)=\frac{\alpha\beta n e^{\frac{-\alpha(n-1)-\alpha}{e^{\beta x}-1}+\beta x}}{\left(e^{\beta x}-1\right)^{2}} \tag{19}$$

VI. PARAMETERS ESTIMATION

This section represents the estimation of parameters of the FETE distribution using the MLE method. Let $x_1, x_2, ...x_n$ be random samples from FETE distribution. Then the likelihood function of FETE distribution is given as

$$l=\prod_{i=1}^{n}\left(\frac{\alpha\beta e^{\beta x-\frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^{2}}\right) \tag{20}$$

$$l=\frac{\alpha^{n}\beta^{n}e^{\beta\sum_{i=1}^{n}x_i-\frac{\alpha}{e^{\beta\sum_{i=1}^{n}x_i}-1}}}{\sum_{i=1}^{n}\left(e^{\beta x}-1\right)^{2}} \tag{21}$$

Taking the log of (21), then the log-likelihood function presented as

$$\log(l)=n\log(\alpha)+n\log(\beta)-\beta\sum_{i=1}^{n}x_i-\frac{\alpha}{e^{\beta\sum_{i=1}^{n}x_i}-1}-2\log\sum_{i=1}^{n}\left(e^{\beta x}-1\right) \tag{22}$$

Therefore, MLE of $\alpha\ and\beta$ which maximize (22) must satisfy the following normal equation.

$$\frac{\partial\log(l)}{\partial\alpha}=\frac{n}{\alpha}-\frac{1}{e^{\beta\sum_{i=1}^{n}x_i}-1} \tag{23}$$

$$\frac{\partial \log(l)}{\partial \beta} = \frac{n}{\beta} - \frac{2\sum_{i=1}^{n} x_i e^{\beta \sum_{i=1}^{n} x_i}}{e^{\beta \sum_{i=1}^{n} x_i} - 1} + \frac{\alpha \sum_{i=1}^{n} x_i e^{\beta \sum_{i=1}^{n} x_i}}{\left(e^{\beta \sum_{i=1}^{n} x_i} - 1\right)^2} + \sum_{i=1}^{n} x_i \qquad (24)$$

Equations (23) and (24) are not in closed form so, it is difficult to estimate the parameters exactly. But still, we can get the estimate of the parameters using some numerical techniques.

V. SHANNON ENTROPY

To measures the variation of the uncertainty of a random variable we need entropy. The greater value of entropy means greater uncertainty and the smaller value indicate small uncertainty in the data. Shannon in 1951 developed an entropy that is defined as

$$SE(x) = -\int_0^\infty f(x) \log f(x) dx \qquad (25)$$

Using (4) in (25), we get

$$SE(x) = -\int_0^\infty \left(\frac{\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} \right) \log \left(\frac{\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} \right) dx \qquad (26)$$

The simplification of (26) is proceeded as

$$\log \left(\frac{\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} \right) = \log \left(\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}} \right) - \log\left(\left(e^{\beta x}-1\right)^2 \right)$$

$$\log(\alpha) + \log(\beta) + \beta x - \frac{\alpha}{e^{\beta x}-1} - 2\log(e^{\beta x}-1)$$

Equation (26) get the form.

$$= -\log(\alpha) \int_0^\infty \frac{\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} dx - \log(\beta) \int_0^\infty \frac{\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} dx -$$

$$\int_0^\infty \frac{\beta x \alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} dx + \int_0^\infty \frac{\alpha}{e^{\beta x}-1} \frac{\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} dx$$

$$+2\int_0^\infty \log\left(e^{\beta x}-1\right) \frac{\alpha\beta e^{\beta x - \frac{\alpha}{e^{\beta x}-1}}}{\left(e^{\beta x}-1\right)^2} dx$$

The Shannon entropy becomes.

$$SE(x) = -\log(\alpha) - \log(\beta) - \left[\sum_{i=1}^{n} \frac{(-1)^k}{k} \alpha^k (-1)^{2k} \Gamma(1-k, -z) + 1 + 2\log(z) e^{-\frac{\alpha}{z}} - \Gamma(0, \alpha/z) \right] \Bigg|_0^\infty \qquad (27)$$

VI. SIMULATION STDUY

In the simulation study, we generate data from FETE distribution by using equation (10). The simulation experiment is repeated 1000 times for the different sample sizes i.e., n=30, 60, 90 and 120. We have estimated Bias and MSE which are shown in Table-2. The results show that as n increases the Bias and MSE decreases.

Table 2: MLE, MSE and Bias for the FETE Estimators

α	β	n	MLE (α)	MLE (β)	MSE (α)	MSE (β)	Bias (α)	Bias (β)
0.3	0.03	30	0.4201	0.0380	0.0845	0.0004	0.1201	0.0080
		60	0.3604	0.0340	0.0337	0.0002	0.0604	0.0040
		90	0.3375	0.0328	0.0170	0.0001	0.0375	0.0028
		120	0.3251	0.0318	0.0103	0.0001	0.0251	0.0018
0.03	0.3	30	0.1696	1.5507	0.0360	2.5469	0.1396	1.2507
		60	0.1396	1.3125	0.0178	1.4931	0.1096	1.0125
		90	0.1249	1.1847	0.0121	1.0329	0.0949	0.8847
		120	0.1197	1.1393	0.0103	0.8954	0.0897	0.8393
0.05	0.4	30	0.1914	1.3910	0.0421	1.8004	0.1414	0.9910
		60	0.1482	1.1230	0.0159	0.8516	0.0982	0.7230
		90	0.1406	1.0691	0.0122	0.6553	0.0906	0.6691
		120	0.1336	1.0194	0.0095	0.5222	0.0836	0.6194

VII. APPLICATIONS

Two real data sets are examined here, to explain how FETE can be a good model for lifetime data compared to many well-known distributions such as the exponential, Weibull [23], Alpha Power Inverted Exponential [24] and Alpha Power Exponential [25] (EXP, W, APIE, APE). We calculated AIC, CAIC, HQIC and BIC for all selected distributions. Table-3 and Table-4 show the goodness-of-fit of the FETE distribution, as the FETE distribution has minimum AIC, CAIC, HQIC and BIC compared to other distributions for both data set.

Date Set 1- This data set shows the losses due to wind catastrophes documented in 1977 used by [26]. There are 40 observations in the data set that were recorded at the nearest \$1,000,000. The value of the data set is as follows (in millions of dollars).

2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 8, 8, 9, 15, 17, 22, 23, 24, 25, 27, 32, 43

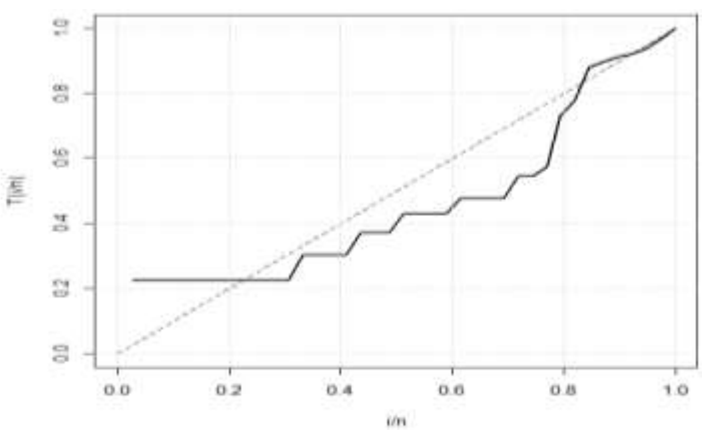

Fig 3: TTT plot of Losses data

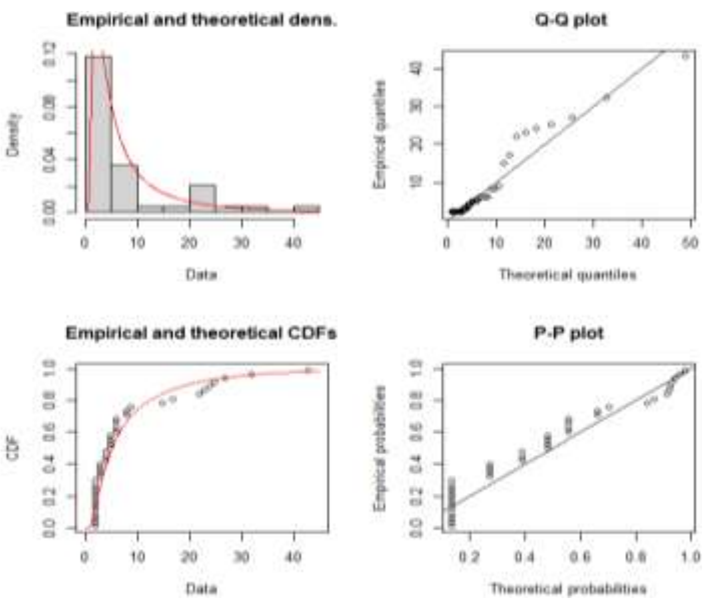

Fig 4: Theoretical, Empirical, QQ & PP Plots of Losses data

Fig 3 shows the TTT for the first data set losses due to wind catastrophes which indicates that the data set follows a non-monotonic hazard rate shape, while Fig 4 demonstrated the theoretical and empirical with P-P and Q-Q plots of the FETE distribution for the first data set those confirmations the best fit of the data on FETE distribution. Table 3 reflects the MLE estimates for the selected distributions and Table 4 shows the goodness of fit measures. Table 4 declared that FETE distribution has a fever value for all the goodness of fit criteria and hence, the proposed model provides a better fit as compared to others.

Table 3: ML Estimates

Models	**Estimates**
FETE	0.27793585, 0.06443095
Exp	0.1130212
W	0.111984, 1.004145
APIE	0.1411834, 5.6512002
APE	1.502518, 0.123764

Table 4: Goodness of Fit Criteria

Model	**AIC**	**CAIC**	**BIC**	**HQIC**
FETE	235.7064	236.0397	239.0335	236.9002
Exp	250.0387	250.1468	251.7022	250.6355
W	252.0375	252.3709	255.3647	253.2313
APIE	235.9599	236.2933	239.287	237.1537
APE	253.0238	253.3571	256.3509	254.2175

Data Set 2- The data set is related to relief times (minutes) consists of 20 patients receiving an analgesic. For more detail see [27]. The observations of the data set are.

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2

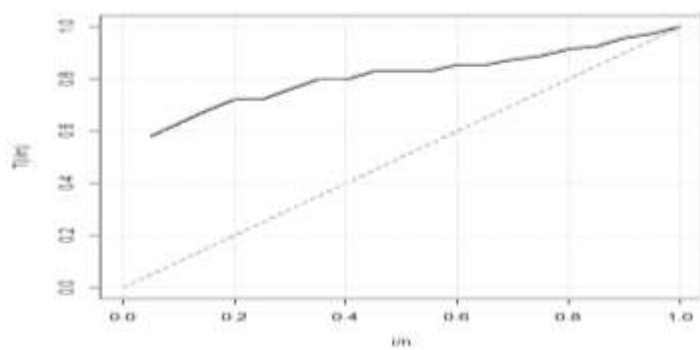

Fig 5: TTT Plot of Relief times data

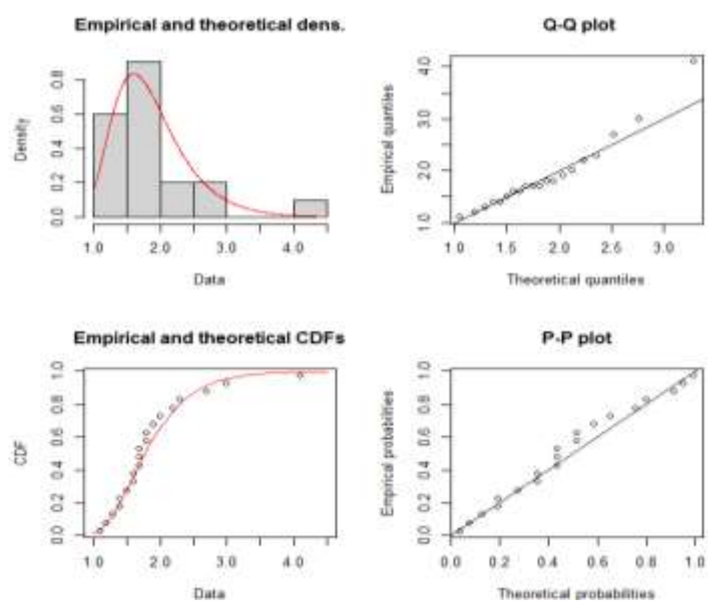

Fig 6: Theoretical, Empirical, QQ & PP Plots of Relief times data

Fig 5 displays that the Relief times of patients' data exhibits the monotonic hazard rate shape. Fig 6 supported the empirical P-P, Q-Q, empirical and theoretical plots of the FETE distribution for the Relief times of patients' data. Tables 5 & 6 includes the MLEs of parameters and measures of the goodness of fit. Table 6 declared that FETE distribution has a fever value for all the goodness of fit criteria and hence, the proposed model provides a better fit as compared to others.

Table 5: Maximum Likelihood Estimates

Model	Estimates
FETE	13.922677, 1.689439
Exp	0.5263803
W	0.1223502, 2.7731621
APIE	0.004579564, 3.189847788
APE	24.273821, 0.9915934

Table 6: Goodness of Fit Criteria

Model	AIC	CAIC	BIC	HQIC
FETE	37.80179	38.50767	39.79326	38.19055
Exp	67.67416	67.89638	68.66989	67.86853
W	45.17476	45.88064	47.16622	45.56351
APIE	49.88164	50.58753	51.87311	237.1537
APE	54.47939	55.18527	56.47086	50.2704

VIII. CONCLUSION

In this paper, different lifetime probability distributions have been used to analyze two different patterns of the real data sets. For both cases, Flexible Exponential type Exponential (FETE) distribution performs better than other probability models. Some statistical properties are obtained (rth moment, Survival function, Hazard function, Quantile function, Median and Mode) in addition with the parameter estimation. Moreover, a simulation study has been conducted to study the consistency and accuracy of the MLE parameters. All in all, it can be claimed that FETE distribution leads a preferable result irrespective of the nature of the data.

REFERENCES

[1] Nadarajah, S. and Kotz, S, The beta exponential distribution. *Reliability engineering & system safety*, 2006, 91(6), pp.689-697.

[2] Gupta, R.D. and Kundu, D. Exponentiated exponential family: an alternative to gamma and Weibull distributions. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, 2001, *43*(1), pp.117-130.

[3] Cordeiro, G. M., Ortega, E. M., & Nadarajah, S. The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, 2010, *347*(8), pp.1399-1429.

[4] Gupta, R. D., & Kundu, D, A new class of weighted exponential distributions. *Statistics*, 2009, *43*(6), pp.621-634.

[5] Tahmasbi, R., & Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. *Computational Statistics & Data Analysis*, 2088, *52*(8), pp.3889-3901.

[6] Barreto-Souza, W., Santos, A.H. and Cordeiro, The beta generalized exponential distribution. *Journal of statistical Computation and Simulation*,2010, *80*(2), pp.159-172.

[7] Ristic, M.M.; Balakrishnan, N. The gamma-exponentiated exponential distribution. J. Stat. Comput. Simul. 2012, 82, pp.1191–1206.

[8] Oguntunde, P.E.; Balogun, O.S.; Okagbue, H.I.; Bishop, S.A. The Weibull-exponential distribution: Its properties and applications. J. Appl. Sci. 2015, 15, pp.1305–1311.

[9] Ristic, M.M.; Kundu, D. Marshall-Olkin generalized exponential distribution. Metron 2015, 73, pp.317–333.

[10] George, R.; Thobias, S. Kumaraswamy Marshall-Olkin Exponential Distribution. Commun. Stat. Theory Methods 2019, 48, pp.1920–1937.

[11] Cordeiro, G.; de Castro, M. A new family of generalized distributions. J. Stat. Comput. Simul. 2011, 81, pp.883– 898.

[12] Haq, M.; Elgarhy, M. The odd Fréchet- G family of probability distributions. J. Stat. Appl. Probab. 2018, 7, pp.189–203.

[13] Zubair, A.; Elgarhy, M.; Hamedani, G.; Butt, N. Odd generalized N-H generated family of distributions with application to exponential model. Pak. J. Stat. Oper. Res. 2020, 16, pp.53–71.

[14] Alzaatreh, A.; Lee, C.; Famoye, F. A new method for generating families of continuous distributions. Metron 2013, 71, pp.63–79.

[15] Badr, M.M.; Elbatal, I.; Jamal, F.; Chesneau, C.; Elgarhy, M. The transmuted odd Fréchet-G family of distributions: Theory and applications. Mathematics 2020, 8, pp.958.

[16] Bourguignon, M.; Silva, R.B.; Cordeiro, G.M. The Weibull-G family of probability distributions. J. Data Sci. 2014, 12, pp.1253–1268.

[17] ElSherpieny, E.A.; Almetwally, E.M. The Exponentiated Generalized Alpha Power Family of Distribution: Properties and Applications. Pak. J. Stat. Oper. Res. 2022, 8, pp.349–367.

[18] Alotaibi, N.; Elbatal, I.; Almetwally, E.M.; Alyami, S.A.; Al-Moisheer, A.S.; Elgarhy, M. Truncated Cauchy Power Weibull-G Class of Distributions: Bayesian and Non-Bayesian Inference Modelling for COVID-19 and Carbon Fiber Data. Mathematics 2022, 10, pp.1565.

[19] Elbatal, I.; Alotaibi, N.; Almetwally, E.M.; Alyami, S.A.; Elgarhy, M. On Odd Perks-G Class of Distributions: Properties, Regression Model, Discretization, Bayesian and Non-Bayesian Estimation, and Applications. Symmetry 2022, 14, pp.883.

[20] Bantan, R.A.; Chesneau, C.; Jamal, F.; Elgarhy, M. On the Analysis of New COVID-19 Cases in Pakistan Using an Exponentiated Version of the M Family of Distributions. Mathematics 2020, 8, 953. doi:10.3390/math8060953.

[21] Tahir, M.H.; Cordeiro, G.M.; Alzaatreh, A.; Mansoor, M.; Zubair, M. The Logistic-X family of distributions and its applications. Commun. Stat.-Theory Methods 2016, 45, pp.7326–7349.

[22] Mahdavi, A.; Kundu, D. A new method for generating distributions with an application to exponential distribution. Commun. Stat.-Theory Methods 2017, 46, pp.6543–6557.

[23] Weibull, W. A statistical distribution function of wide applicability. *Journal of applied mechanics*, 1951, *18*(3), pp. 293-297.

[24] Ceren, Ü. N. A. L., Cakmakyapan, S., & Gamze, Ö. Z. E. L, Alpha power inverted exponential distribution: Properties and application. *Gazi University Journal of Science*, 2018, *31*(3), pp.954-965.

[25] Mahdavi, A., & Kundu, D, A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, 2017, *46*(13), pp.6543-6557.

[26] Hogg, R. V., & Klugman, S. A, *Loss distributions*, John Wiley & Sons, 2009, 249.

[27] Gross, A. J., & Clark, V, Survival *distributions: reliability applications in the biomedical sciences*, John Wiley & Sons, 1975.

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