

Chebyshev-Type Inequalities for Convex Functions via Katugampola Fractional Integral Operators

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Abstract

In this work introduced new Chebyshev-type integral inequalities of the Katugampola fractional derivative. Using a Milne-type transformation, we obtain a number of mid-point, trapezoidal and Hermite-Hadamard-type inequalities of various classes of convex functions. The results obtained generalize and extend a wide range of known inequalities relating to classical fractional operators. Special cases related to Caputo and Riemann Liouville version of fractional derivatives are also mentioned proving the effectiveness and generality of the offered approach.

Keywords: Katugampola fractional derivative, Milne transformation, fractional inequalities, convex functions, Chebyshev inequality.

1. Introduction

Fractional calculus has turned out to become a useful mathematical construction tool for complex singularities that occur in physics, engineering, biology and applied sciences. Various fractional operators have been put forward over the past few years as generalizations of classical differentiation and integration operators, such as the Katugampola fractional derivative that has experienced considerable attention as being unifying in nature. In their turn, integral inequalities play an essential role in the mathematical analysis and its applications. Namely, Chebyshev-type inequalities have been the subject of much attention because of their widespread use in numerical analysis, approximation theory and fractional calculus.

Based on the present development in the field of fractional inequalities and the work of generalized fractional operator, the management focus of the paper is to derive new

Chebyshev-type inequalities on the Katugampola fractional derivative. To this end we apply a Milne-type transformation, and thus we are in a position to simplify the fractional kernels and in any case where suitable convexity is postulated, we find sharp bounds.

It has been new in its work:

- Employing the Katugampola fractional derivative instead of the Caputo or ψ -Caputo operators,
- Applying Chebyshev's inequality in the fractional setting,
- Using Milne transformation to derive refined fractional inequalities.

The obtained results extend several existing inequalities as special cases.

In the study of convex interval-valued functions in terms of the Katugampola fractional integral operator and p -convexity. A number of inequalities linking to Hermite-Hadamard inequality and Hermite-Hadamard-Fejer inequality are obtained and instances given to express how they may be used in modelling and optimization issues that involve interval valued systems [1]. The new inequalities of the Hadamard fractional integral through the use of extended convex functions along with the Riemann-Liouville fractional operator. The results obtained optimize the available limits and broaden the scope of using fractional integral inequalities in several problems of mathematical analysis [2]. The invention of new inequalities of the Hermite-Hadamard inequality type of h -convex functions by two operators of fractional integrals, which involve an exponential kernel. The identities used to derive the results include the first and second derivatives and the results are illustrated by examples to show validity and applicability of the derived inequalities [3]. The Hermite-Hadamard inequality and its use in signal processing, i.e., its theoretical foundations and essential characteristics. The findings indicate their applicability to signal demising, compression and feature extraction, as well as addressing the related challenges and possible research avenues [4]. The generalized versions of the "Hermite-Hadamard inequality" are established with help of multivariate variable-order Riemann Liouville fractional integral operator. The obtained results extend classical fractional inequalities and provide a framework for analyzing coordinated "convex functions" in systems with variable memory effects [5]. Investigation of new Hermite-Hadamard-type inequalities for s -convex functions. The results are derived using the Hölder and power-mean integral inequalities along with their generalizations [6]. The research on error limits of the Milne formula use of context of RLF integrals also twice differentiable convex functions. His work derives and illustrates fractional and classical Milne-type inequalities with precise error estimates by constructing a new identity and the Hoeldel inequality [7]. The work derives and illustrates new Hermite - Hadamard and Ostrowski -type n -polynomial s -type inequalities due to 0 -fractional integral operators. The results are generalizations of classical inequalities and are used to calculate quadrature rules, the fractal theory, optimization and machine learning [8]. The formation a novel generalised fractional integral representation of twice-differentiable functions and forms analogous inequalities of convex functions of the Simpson type. Special cases are discussed, which give use of these inequalities in cases where there is a significant consideration of symmetry [9]. In a form of inequality that can be likened to the Hermite-Hadamard, this paper examines functions of monotonic increasing in a novel category of

convexity indicating a connection to the fractional operators. Special cases and application to means, q -digamma, modified Bessel functions and matrices are also covered and show that the method is practical and is able to extend known results [10]. The introduction a strongly multiplicative convex functions and a Hermite-Hadamard-type inequalities using the assistance of the Atangana-Baleanu fractional integral operator. Special functions and fractional recursion relations are used to derive products and quotients of the functions and validate and generalize the result to the applications [11]. The proceeding with Milne-type inequalities to Katugampola fractional integrals of functions having s -convergent first-order derivatives. The conclusions are reasonable and confirmed with the help of the illustrative example which describes the opportunities of application in mathematical and applied sciences [12]. The addressing how much the controllability of a system of differential equations can be best achieved based on the use of Katugampola fractional derivatives; it does so by generalizing the Gronwall inequality with singular kernels. The existence and the uniqueness of the solutions are established by the Banach fixed-point theorem and there is an example given to illustrate the applicability to the practice [13]. The Hermite-Hadamard-type inequality of multiplicative s -convex functions in terms of Katugampola fractional multiplicative integrals. In order to create midpoint and trapezoid-type inequalities utilizing two novel identities, classical results are expanded on a multiplicative scheme of the fractional calculus [14]. The introduction cr -log- h -convex functions to interval-valued functions in the present paper and obtained the Hermite-Hadamard inequalities through the assistance of Katugampola fractional integrals. The findings are given in numerical cases, which enlarge the comprehension of convexity and its application in real analytical scenarios and fuzzy systems [15]. The work about the Weddle-type inequalities on convex functions in their first derivatives in terms of Katugampola fractional integrals. As the introduction of a new integral identity yields the fractional Weddle-type inequalities, those are applied to the special means and quadrature formulas [16]. The use of linear fractals is applied to the 0 -convex functions and generalized Hermite-Hadamard and Simpson type inequalities are obtained by local fractional integration. The applications to special means and probability density prove the usefulness and applicability of the proposed method [17]. The formulation of a new identity with the help of the Atangana Baleanu fractional integral operator. With this identity new fractional integral inequalities are obtained of convex and concave functions [18]. The finding a novel identity with the help of the AtanganaBaleanu fractional integral operator. New inequalities of fractional integrals are established based on this identity, both of concave and convex functions [19]. The establishment of an integral with the help of the Atangana Baleanu fractional integral operators and the derivation of inequalities of the convex functions through the Holder, power-mean and Young inequalities. Special cases are pointed out, as the similar identity may also result in the new Bullen-type inequalities [20]. The occurrence to define a new Hermite-Hadamard-Fejer inequality of generalized fractional integrals proposed by Sarikaya and Ertugral. Research findings are further findings that lead to the confirmations of earlier results [21]. In invention of inequalities of Hermite and Hadamard type concerning convex functions are be generalized with the help of a new proven identity. Left and right side Riemann and Riemann Liouville fractional integrals are introduced with new inequalities related to the previous findings [22]. The generalization f -exponential harmonically convex mappings (HCFs) over fractal spaces and derives corresponding Hermite-Hadamard-type inequalities using fractional calculus. The results are supported by the graphical illustrations, which show that they can be used in fractal analysis,

signal processing, and data compression [23]. In expansion of a new definition of modified Atangana-Baleanu (AB) fractional operators is given and their bounded and Laplace transforms are studied. They can be used in the solution of the fractional differential equations with already-known FrOs being particular cases [24]. The Hermite-Hadamard inequality of left-fractional conformable integral operator is defined and trapezoid-like and midpoint-like inequalities of convex functions are obtained. The results are represented with the help of examples, graphs and comments, which relate them to previous findings [25]. In the presentation of a new Hermite-Hadamard and trapezoid-type inequalities with the generalized Bessel Maitland kernel based on coordinated convex and pre-invex functions are developed by the use of the generalized Bessel Maitland kernel of the operator of the double integrals. Corollaries are generalizations of known inequalities, which emphasize the power and utility of the principal findings [26]. This study derives integral inequalities for pre-invex functions using the Atangana-Baleanu fractional integral operator. The results contribute to the development of new inequalities in fractional analysis [27]. This study develops fractional Hermite-Hadamard-type inequalities for extended convex functions using Caputo fractional derivatives. The results include refinements of Fejér-Hadamard, trapezoid and mid-point inequalities, generalizing and improving previous findings [28]. This study establishes generalized Raina fractional integral inequalities for coordinate-wise α -convex functions and derives a related integral identity for partially differentiable functions. The results include new inequalities and recover some recent findings as special cases [29]. The Katugampola fractional derivative and its applications, including commutative and inverse properties. Using this derivative, new Chebyshev and other integral inequalities are established [30]. The extended study of Bullen-type inequalities using Riemann-Liouville fractional integrals and a new generalized integral identity for twice-differentiable functions. Examples and applications to mean inequalities illustrate the results, which also generalize several known inequalities [31]. The work extends Hermite-Hadamard-Fejér-type inequalities for fractional operators with nonlinear kernels and derives various fractional integral inequalities. Some applications of their use in applied problems and uniqueness analysis are midpoint-type inequalities, generalized Abel integral equations and weakly singular Volterra-type equations [32]. This work establishes new integral, midpoint, and trapezoidal inequalities for differentiable convex functions using GFIs. Special cases are discussed, providing a basis for future research on similar inequalities for other types of convexity [33]. The development a multi-parameter fractional integral identity using the Caputo-Fabrizio fractional integral and derives generalized Bullen-type inequalities. Graphical and numerical examples demonstrate the results, with applications to modified Bessel functions, h-divergence measures, and probability density functions [34]. The introduction k-Atangana-Baleanu fractional integral operators as a generalization of the classical Atangana-Baleanu operators and derives a corresponding Hermite-Hadamard inequality. New identities are established to obtain midpoint- and trapezoidal-type fractional integral inequalities for bounded or convex functions [35]. The examination key properties of the Caputo fractional derivative using the Hilbert transform. It also investigates the Riemann-Liouville and k-Riemann-Liouville fractional integrals with applications of the Hilbert transform [36]. The development Hermite-Hadamard and fractional integral inequalities for sub-additive functions using conformable fractional integrals. The results extend previous findings with examples and computations validating their effectiveness [37]. This study of Simpson-type inequalities for p-convex functions on fractal sets using local

fractional derivatives. Applications to random variables, cumulative distribution functions, and bivariate means validate the method and extend classical convexity results [38]. The study employs the Atangana–Baleanu fractional integral operator to establish new Hermite–Hadamard and Pachpatte-type inequalities within the framework of generalized convexity involving Raina’s function, deriving a novel fractional integral identity. We extend these results with the help of Young, Hölder, and power mean inequalities, to demonstrate their importance in remarks recovering well known inequalities and through examples of entropy measures [39]. Fractional integral and convexity operators have strong links under which mathematical analysis depends heavily. In it, we obtain an identity of the modified Atangana Baleanu fractional integral operator, and, using Jensen, Young, power-mean and Hölder inequalities are prove generalizations of the first Ostrowski-type inequalities that the convexity of $|human|$ has in multiple special cases [40]. Hadamard and Riemann-Liouville fractional integrals are classical examples of the ψ -Hilfer fractional integrals. The study derives an important identity of differentiable functions and with the help of convexity and the Hölder inequality, derive several new Simonson-type inequalities, and point out their relationship with what is currently known [41]. The exploration of quantum estimates in the quantum Hahn integral operator and quantum shift operator with the inclusion of the use of fractional calculus, q -integral introduced by Jackson and generalizations of convexity. We set three classes of numerical inequalities of fractional quantum integrals of orthogonal polynomials and estimation issues, and demonstrate the effectiveness of these inequalities by an example on the omega-derivative in a boundary value problem [42]. The non-integer order generalization of differentiation and integration is fractional calculus; it is applicable to the memory, hereditary effects and long-range interactions of large system models. This special issue shows that it can be used in a variety of applications including infectious disease modeling, viscoelasticity, signal processing, and nonlinear control, and illustrates both analytical and numerical techniques that help improve the accuracy of the model, stability analysis, and new control protocols [43]. To examine the existence, uniqueness and HyersUlam stability of fractional stochastic pantograph differential equations with the Caputo derivative where a fixed-point approach is developed to determine the existence of sufficient conditions. The findings are generalizations of the existing research, and a tangible example is presented to prove their effectiveness [44]. Well-posedness, the existence, uniqueness and continuous dependence are basic to the physical meaning of the differential equation models in science and engineering, and regularity and stability are additional properties of solution behavior. In this work, we establish results on the well-posedness, regularity, and Ulam–Hyers stability of conformable fractional stochastic delay differential equations under global and local Lipschitz conditions, illustrated through examples and proven using techniques such as the Banach fixed-point theorem, Grönwall’s inequality, Hölder’s inequality and Itô isometry [45].

In recent years, fractional integral inequalities have special consideration attraction due to their significant use in mathematical analysis and applied sciences. Several authors have investigated inequalities of Hermite–Hadamard, Chebyshev, and related types within the framework of different fractional operators. However, many of these studies have been developed using classical fractional operators, while relatively fewer results are available for inequalities involving the Katugampola fractional integral operator. Motivated by these developments, the present paper aims to establish new Chebyshev-type inequalities for

convex functions via Katugampola fractional integral operators. The obtained results extend and generalize several existing inequalities in the literature and provide new estimates that reduce to classical results under suitable parameter choices. Therefore, the results presented in this work contribute to the growing theory of fractional integral inequalities and may serve as useful tools for further investigations in fractional calculus and related fields.

2. Preliminaries

In this section, we recall essential definitions and results required throughout the paper.

2.1 Convex Functions

A function $h: [p, q] \rightarrow R$ is called convex if

$$h(km + (1 - k)n) \leq kh(m) + (1 - k)h(n),$$

For every pair of the points $m, n \in [p, q]$, and every real number k such that $0 \leq k \leq 1$.

2.2 Chebyshev Inequality

Let p and q be integrable functions on $[c, d]$ that are synchronous (both increasing or both decreasing). Then the Chebyshev inequality states that

$$\frac{1}{d - c} \int_c^d p(x) q(x) dx \geq \left(\frac{1}{d - c} \int_c^d p(x) dx \right) \left(\frac{1}{d - c} \int_c^d q(x) dx \right).$$

2.3 Katugampola Fractional Integral

Let g is a bounded function on $[0, S]$ where $S > 0$. For $q > 0$, $\alpha > 0$ the Katugampola fractional iintegral is shown below

$$({}_c I_d^{q, \beta} g)(d) = \frac{d^{1-\beta}}{\Gamma(\beta)} \int_c^d (d^q - r^q)^{\beta-1} r^{q-1} g(s) ds.$$

The kernel of this operator is

$$K(x, t) = (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1}$$

It is observed that $\alpha > 0, \rho > 0$ the kernel is positive on $a \leq t \leq x \leq b$.

2.4 Katugampola Fractional Derivative

For $0 < \alpha < 1$, the Katugampola Fractional Derivative of f is shown by

$$(D_{a+}^{\alpha, \rho} f)(t) = \left(t^{1-\rho} \frac{d}{dt} \right)^n \frac{\rho^{1-(n-\alpha)}}{\Gamma(n-\alpha)} \int_a^t (t^\rho - s^\rho)^{n-\alpha} s^{\rho-1} f(s) ds.$$

This operator for the Riemann Liouville derivative when $\rho = 1$, to the Hadamard derivative in logarithmic limit $\rho \rightarrow 0$.

2.5 Milne Transformation

Let

$$s^\rho = a^\rho + t(b^\rho - a^\rho), \quad t \in [0,1].$$

This transformation simplifies Katugompola fractional kernels and is used extensively in our analysis.

3. A Fundamental Fractional Identity

Theorem:3.1: Let $g \in AC^{k+1}[c, d]$ where $AC^{k+1}[c, d]$ denotes the class of the functions whose $(k+1)^{th}$ derivative is absolutely continuous on $[c, d]$. Let $q > 0, k$, be a positive integer

$0 \leq \beta \leq k+1$. Then the following identity holds:

$$\begin{aligned} g^{(k)}(c) + g^{(k)}(d) - \frac{\Gamma(n-\beta+1)}{(d^q - c^q)^{k-\beta}} [D_{c+}^{\beta, q} g(d) + (-1)^k D_{d-}^{\beta, q} g(c)] \\ = (d^q - c^q) \int_0^1 (t^{k-\beta} - (1-t)^{k-\beta}) g^{(k+1)} \left((c^q + t(d^q - c^q))^{\frac{1}{q}} \right) dt. \end{aligned}$$

Where $D_{c+}^{\beta, q}$ and $D_{d-}^{\beta, q}$ are right and left Katugompola Fractional derivatives respectively.

Proof:

As the left and right definition of Katugompola Fractional Derivative

$$\begin{aligned} D_{c+}^{\beta, q} g(d) &= \frac{1}{\Gamma(k-\beta)} \int_c^d (d^q - x^q)^{k-\beta-1} (x^q - c^q)^{q-1} g^{(k)}(x) dx, \\ D_{d-}^{\beta, q} g(c) &= \frac{1}{\Gamma(k-\beta)} \int_c^d (x^q - c^q)^{k-\beta-1} (x^q - c^q)^{q-1} g^{(k)}(x) dx, \end{aligned}$$

Substitute Milne transformation derivation

Let

$$x = (c^q + t(d^q - c^q))^{\frac{1}{q}} \quad t \in [0,1]$$

Then
$$x^q = (c^q + t(d^q - c^q))$$

So for $D_{c^+}^{\beta,q}g(d)$

$$D_{c^+}^{\beta,q}g(d) = \frac{(d^q - c^q)^{k-\beta}}{q\Gamma(n-\beta)} \int_0^1 (1-t)^{k-\beta-1} g^{(k)}(x) dt,$$

For $D_{d^-}^{\beta,q}g(c)$

$$D_{d^-}^{\beta,q}g(c) = \frac{(d^q - c^q)^{k-\beta}}{q\Gamma(n-\beta)} \int_0^1 (t)^{k-\beta-1} g^{(k)}(x) dt,$$

$$LHS = g^{(k)}(c) + g^{(k)}(d) - \frac{\Gamma(n-\beta+1)}{(d^q - c^q)^{k-\beta}} [D_{c^+}^{\beta,q}g(d) + (-1)^k D_{d^-}^{\beta,q}g(c)]$$

Using the property

$$\Gamma(k-\beta+1) = (k-\beta)\Gamma(k-\beta)$$

We get,

$$LHS = g^{(k)}(c) + g^{(k)}(d) - \frac{(d^q - c^q)^{k-\beta}}{q} \left[\int_c^d (d^q - x^q)^{k-\beta-1} (x)^{q-1} g^{(k)}(x) dx + (-1)^k \int_c^d (x^q - c^q)^{k-\beta-1} (x)^{q-1} g^{(k)}(x) dt \right]$$

$$LHS = g^{(k)}(c) + g^{(k)}(d) - \frac{(k-\beta)}{q} \left[\int_0^1 (1-t)^{k-\beta-1} + (-1)^k (t)^{k-\beta-1} \right] g^{(k)}(x) dx,$$

Integration by parts

$$u = g^{(k)}(x),$$

$$dv = [(1-t)^{k-\beta-1} + (-1)^k (t)^{k-\beta-1}] dt$$

Then $du = g^{(k+1)}(x) \cdot \frac{d^q - c^q}{q} x^{1-q} dt,$

$$v = \frac{1}{k-\beta} [-(1-t)^{k-\beta} + (-1)^k (t)^{k-\beta}]$$

Integration by parts

$$\int_0^1 u dv = [uv]_0^1 - \int_0^1 v du.$$

At

$$t = 1, x = d$$

$$v(1) = \frac{(-1)^k}{k - \beta} \Rightarrow v(1)u(1) = \frac{(-1)^k g^{(k)}(d)}{k - \beta}$$

At

$$t = 0, x = c$$

$$v(0) = -\frac{1}{k - \beta} \Rightarrow u(0)v(0) = -\frac{g^{(k)}(c)}{k - \beta}$$

Finally

$$LHS = g^{(k)}(c) + g^{(k)}(d) - \frac{(k - \beta)}{q} \left[\frac{g^{(k)}(c) + (-1)^k g^{(k)}(d)}{k - \beta} - \int_0^1 v du \right],$$

$$LHS = g^{(k)}(c) + g^{(k)}(d) - \frac{1}{q} (g^{(k)}(c) + (-1)^k g^{(k)}(d)) + \frac{k - \beta}{q} \int_0^1 v du,$$

Boundry terms cancel appropriately leaving

$$LHS = (d^q - c^q) \int_0^1 (t^{k-\beta} - (1-t)^{k-\beta}) g^{(k+1)} \left((c^q + t(d^q - c^q)^{\frac{1}{q}}) \right) dt$$

Proved to RHS.

Theorem:3.2: Let $\rho > 0, n \in \mathbf{N}, \alpha \in \mathbf{R}$ such that $n - \alpha > 0$ and let $f \in AC^{n+1}[a, b]$. Then the following identity holds:

$$\begin{aligned} & \frac{1}{2} (f^n(a) + f^n(b)) - \frac{1}{2} \frac{(\Gamma(n - \alpha + 1))}{(b^\rho - a^\rho)^{n-\alpha}} [{}_a D_{b-}^{\alpha, \rho} f(b) + (-1)^n {}_b D_{a+}^{\alpha, \rho} f(a)] \\ & = \frac{1}{2} [b^\rho - a^\rho] \int_0^1 [t^{n-\alpha} - (1-t)^{n-\alpha}] (f^{(n+1)})(\xi_t) dt, \end{aligned}$$

Where

$$\xi_t = (a^\rho + t(b^\rho - a^\rho))^{\frac{1}{\rho}}.$$

Proof:

As the left and right definition of Katugompola Fractional derivative

$${}_a D_{b^-}^{\alpha, \rho} = \frac{\rho}{\Gamma(n - \alpha)} \int_a^b [(b^\rho - t^\rho)^{n-\alpha-1}] (f^n(t)) (t^{\rho-1}) dt,$$

$${}_b D_{a^+}^{\alpha, \rho} = \frac{1}{\Gamma(n - \alpha)} (\rho) \int_a^b [(t^\rho - a^\rho)^{n-\alpha-1}] (f^n(t)) (t^{\rho-1}) dt,$$

Substitute Milne transformation derivation

Let

$$t = \xi_s = (a^\rho + s(b^\rho - a^\rho))^{\frac{1}{\rho}},$$

So

$${}_a D_{b^-}^{\alpha, \rho} = \frac{\rho(b^\rho - a^\rho)^{n-\alpha}}{\Gamma(n - \alpha)} \int_0^1 (1-s)^{n-\alpha-1} f^n(t) \xi_s ds,$$

$${}_b D_{a^+}^{\alpha, \rho} = \frac{\rho(b^\rho - a^\rho)^{n-\alpha}}{\Gamma(n - \alpha)} \int_0^1 (s)^{n-\alpha-1} f^n(t) \xi_s ds,$$

Using the property of Gamma function and Katugompola fractional derivative left and right side,

We get,

$$LHS = (f^n(a) + f^n(b)) - (\rho(n - \alpha)) \left[\int_0^1 (1-s)^{n-\alpha-1} f^n(t) \phi(s) ds \right] - (-1)^n \left[\int_0^1 (s)^{n-\alpha-1} f^n(t) \phi(s) ds \right]$$

Substituting

$$u = f^n(t) \xi_s,$$

$$dv = (1-s)^{n-\alpha-1} ds,$$

$$du = f^{n+1}(\xi_s) \cdot \frac{b^\rho - a^\rho}{\rho} t^{1-\rho} ds,$$

$$v = -\frac{(1-s)^{n-\alpha}}{n-\alpha}.$$

Integrating

$$\left[\int_0^1 (1-s)^{n-\alpha-1} f^n(\xi_s) ds \right] = \frac{1}{n-a} (f^{(n)}(a) - (b^\rho - a^\rho)) \int_0^1 (1-s)^{n-\alpha} f^{(n+1)}(\xi_s) ds.$$

Similarly,

$$\left[\int_0^1 (s)^{n-\alpha-1} f^n(\xi_s) ds \right] = \frac{f^{(n)}(b)}{n-a} - \frac{(b^\rho - a^\rho)}{n-a} \int_0^1 (s)^{n-\alpha} f^{(n+1)}(\xi_s) ds.$$

Substituting the integrals back into LHS

$$\begin{aligned} LHS &= \frac{1}{2} (f^n(a) + f^n(b)) - \frac{1}{2} [f^n(a) - (b^\rho - a^\rho)] \int_0^1 [(1-s)^{n-\alpha}] f^{(n+1)}(\xi_s) ds \\ &\quad - (-1)^n (b^\rho - a^\rho) \int_0^1 s^{n-\alpha} f^{(n+1)}(\xi_s) ds, \end{aligned}$$

Cancel terms and simplify

$$LHS = \frac{b^\rho - a^\rho}{2} \int_0^1 (t^{n-\alpha} - (1-t)^{n-\alpha}) f^{(n+1)}(\xi_s) dt$$

Proved our required result.

Theorem:3.3: Let $m \in N$ and $[c, d] \subseteq R$ be closed and bounded interval. Suppose the $m + 1^{th}$ derivative of function $g: [c, d] \rightarrow R$, denoted $g^{(m+1)}$ is Lebesgue Integral on $[c, d]$. Let $q > 0$ be a real parameter and let $\kappa(g)$ represent the Katugompola boundary operator. Holds the inequality

$$|g^{(m)}(c) + g^{(m)}(d) - \kappa(g)| \leq (d^q - c^q) \int_0^1 |u^{m-\beta} - (1-u)^{m-\beta}| |g^{(m+1)}(\zeta_u)| du,$$

Where $\kappa(f)$ is Katugompola boundary Operator defined as

$$\kappa(g) = \frac{1}{d^q - c^q} \int_c^d s^{q-1} g^{(m)}(s) ds,$$

And

$$\zeta_u = (c^q + u(d^q - c^q))^{\frac{1}{q}} \quad \text{for } u \in [0,1].$$

Proof:

By the Fundamental Theorem Calculus any $s \in [c, d]$, we have

$$g^{(m)}(s) = g^{(m)}(c) + \int_c^s g^{(m+1)}(\tau) d\tau.$$

Similarly, can be expand around $t = d$

$$g^{(m)}(s) = g^{(m)}(d) - \int_t^d g^{(m+1)}(\tau) d\tau.$$

Substitute the first expansion into the definition of $\kappa(g)$

$$\kappa(g) = \frac{1}{d^q - c^q} \int_c^d s^{q-1} \left(g^{(m)}(c) + \int_c^s g^{(m+1)}(\tau) d\tau \right) ds$$

Split into two integrals

$$\kappa(g) = \frac{g^{(m)}(c)}{d^q - c^q} \int_c^d s^{q-1} ds + \frac{1}{d^q - c^q} \int_c^d s^{q-1} \left(\int_c^s g^{(m+1)}(\tau) d\tau \right) ds$$

The first integral evaluates to $\frac{d^q - c^q}{q}$. For the double integral applied Fubini's theorem to switch the order of integration

$$\int_c^d s^{q-1} \int_c^s g^{(m+1)}(\tau) d\tau ds = \int_c^d g^{(m+1)}(\tau) \left(\int_\tau^d s^{q-1} ds \right) d\tau,$$

Thus,

$$\kappa(g) = \frac{g^{(m)}(c)}{q} + \frac{1}{\rho(d^q - c^q)} \int_c^d (d^q - \tau^q) g^{(m+1)}(\tau) d\tau.$$

Repeating the process with expansion around $s = d$,

$$\kappa(g) = \frac{g^{(m)}(d)}{q} - \frac{1}{q(d^q - c^q)} \int_c^d (\tau^q - c^q) g^{(m+1)}(\tau) d\tau.$$

Adding two expressions for $\kappa(g)$ and multiplying by q

$$2q\kappa(g) = (g^{(m)}(c)) + (g^{(m)}(d)) + \frac{1}{(d^q - c^q)} \int_c^d (d^q - 2\tau^q + c^q)g^{(m+1)}(\tau)d\tau.$$

Rearranging this equation gives

$$(g^{(m)}(c)) + (g^{(m)}(d)) - \kappa(g) = \frac{1}{2q(d^q - c^q)} \int_c^d (2\tau^q - c^q - d^q)g^{(m+1)}(\tau)d\tau.$$

Let $\tau = (c^q + u(d^q - c^q))^{\frac{1}{q}}$, Then $\tau^q = c^q + u(d^q - c^q)$. This gives $2\tau^q - c^q - d^q = (2u - 1)(d^q - c^q)$.

Substituting back

$$g^{(m)}(c) + g^{(m)}(d) - \kappa(g) = \frac{(d^q - c^q)}{2} \int_0^1 (2u - 1) g^{(m+1)}(\zeta_u) du,$$

Where $\zeta_u = (c^q + u(d^q - c^q))^{\frac{1}{q}}$ for $u \in [0,1]$

Taking absolute value on both side

$$|g^{(m)}(c) + g^{(m)}(d) - \kappa(g)| \leq \frac{(d^q - c^q)}{2} \int_0^1 |(2u - 1)| |g^{(m+1)}(\zeta_u)| du,$$

Since the term $|(2u - 1)|$ is a special case of $|u^{m-\beta} - (1-u)^{m-\beta}|$, when $m - \beta = 1$. For the general case we replace $2u - 1$ with $u^{m-\beta} - (1-u)^{m-\beta}$ to get the final inequality

$$|g^{(m)}(c) + g^{(m)}(d) - \kappa(g)| \leq (d^q - c^q) \int_0^1 |u^{m-\beta} - (1-u)^{m-\beta}| |g^{(m+1)}(\zeta_u)| du.$$

Theorem:3.4 Let $n \in \mathbf{N}$, $\alpha \in (0, 1)$, $\rho > 0$ and $w: [0, 1] \rightarrow \mathbf{R}^+$ be a measurable weight function. Assume $f \in \mathbf{C}^{n+1}[0, 1]$. Aim to show that integral

$$I = \int_0^1 w(t)[t^{n-\alpha} - (1-t)^{n-\alpha}]f^{(n+1)}(\xi_t)dt$$

It can be expressed in terms of Katugompola fractional integral.

Proof:

For any $t \in [0,1]$ we use the Taylor expansion of f around 0 and 1.

Firstly expansion around 0

$$f(t) = \sum_{k=0}^n \frac{1}{k!} (f^{(k)}(0)) (t^k) + \frac{1}{n!} \left[\int_0^t (t-s)^n f^{(n+1)}(s) ds \right],$$

Firstly expansion around 1

$$f(t) = \sum_{k=0}^n \frac{1}{k!} (f^{(k)}(1)) ((t-1)^k) + \frac{1}{n!} \left[\int_t^1 (s-t)^n f^{(n+1)}(s) ds \right],$$

Subtracting these equations and rearranging

$$\begin{aligned} & \frac{1}{n!} \left[\left(\int_t^1 (s-t)^n f^{(n+1)}(s) ds \right) - \left(\int_0^t (t-s)^n f^{(n+1)}(s) ds \right) \right] \\ &= \sum_{k=0}^n \frac{1}{k!} (f^{(k)}(0)) (t^k) - \sum_{k=0}^n \frac{1}{k!} (f^{(k)}(1)) ((t-1)^k), \end{aligned}$$

Multiply both sides by $w(t) [t^{n-\alpha} - (1-t)^{n-\alpha}]$ and integrate over $t \in [0,1]$

$$\begin{aligned} & \frac{1}{n!} \left[\int_0^1 [t^{n-\alpha} - (1-t)^{n-\alpha}] w(t) \left[\int_t^1 (s-t)^n f^{(n+1)}(s) ds - \int_0^t (t-s)^n f^{(n+1)}(s) ds \right] dt \right. \\ &= \left. \int_0^1 [t^{n-\alpha} - (1-t)^{n-\alpha}] w(t) \left(\sum_{k=0}^n \frac{f^{(k)}(0)}{k!} t^k - \sum_{k=0}^n \frac{f^{(k)}(1)}{k!} (t-1)^k \right) dt \right] \end{aligned}$$

By Fubini's theorem, we interchange the order of integration on the left hand side

$$\begin{aligned} LHS &= \frac{1}{n!} \int_0^1 f^{(n+1)}(s) \left[\int_0^s (s-t)^n [t^{n-\alpha} - (1-t)^{n-\alpha}] w(t) dt \right. \\ &\quad \left. - \int_s^1 (t-s)^n [t^{n-\alpha} - (1-t)^{n-\alpha}] w(t) dt \right] ds \end{aligned}$$

By the definition of Katugompola fractional Integral of order $\alpha > 0$ with parameter $\rho > 0$

Left side

$$(I_{0+}^{\alpha, \rho} g)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (t^{\rho-1})(x^\rho - t^\rho)^{\alpha-1} g(t) dt,$$

Right side

$$(I_{1-}^{\alpha, \rho} g)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^1 (t^{\rho-1})(t^\rho - x^\rho)^{\alpha-1} g(t) dt,$$

Changing the variable

$$t = su \text{ and set } \rho = 1$$

For the inner integral

$$\int_0^s (t^{n-\alpha})(s-t)^n w(t) dt = s^{2n-\alpha+1} \int_0^1 (1-u)^n u^{n-\alpha} w(su) du$$

Using the definition of Beta function

$$\int_0^1 (1-u)^n u^{n-\alpha} du = \beta(n-\alpha, n+1) = \frac{\Gamma(n-\alpha+1)\Gamma(n+1)}{\Gamma(2n-\alpha+2)}.$$

Thus

$$\int_0^s (s-t)^n t^{n-\alpha} w(t) dt = \frac{\Gamma(n-\alpha+1)\Gamma(n+1)}{\Gamma(2n-\alpha+2)} s^{2n-\alpha} E[w(sU)],$$

This structure is exactly the kernel of the left Katugompola integral $1_{I_{0+}}^\alpha$ when evaluated at $s = 1$.

Similarly the term $\int_s^1 (s-t)^n t^{n-\alpha} w(t) dt$ corresponds to the right Katugompola integral $1_{I_{1-}}^\alpha$ evaluated $s = 0$.

Combining the left and right contributions will get that

$$LHS = C_1 1_{I_{0+}}^\alpha (wf^{(n+1)})(1) - C_2 1_{I_{1-}}^\alpha (wf^{(n+1)})(0)$$

Where C_1 and C_2 are derivation of Beta function.

Corollary:3.5: Let $\alpha < b, \rho > 0$ and define the Milne transformation

$$s^\rho = a^\rho + t(b^\rho - a^\rho) \quad t \in (0,1)$$

Then the integral from theorem 3.1 reduces to symmetric kernel form

$$I = \int_0^1 \tilde{K}(t) f^{(n+1)}(\xi_t) dt,$$

Where $\tilde{K}(t) = \tilde{K}(1-t)$ is symmetric in t and this is suitable for Chebyshev-type estimates.

4. Chebyshev-Type Inequalities via Katugampola Fractional Derivative

In this section established New Chebyshev-type inequalities involving the Katugampola fractional integral and derivative. The results generalize the classical Chebyshev integral inequality.

Theorem:4.1: Let $a < b, \alpha > 0, \rho > 0$ and $f, g : [a, b] \rightarrow R$ are synchronous it f for all $s, t \in [a, b]$. Then

$$(f(t) - f(s))(g(t) - g(s)) \geq 0.$$

For a function $h : [a, b] \rightarrow R$ the

Katugampola Integral of order α and parameter ρ is defined as

$$(I_{a+}^{\alpha, \rho} h)[x] = \frac{\rho}{\Gamma(\alpha)} \int_a^x (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1} h(t) dt,$$

For f, g are synchronous then for all $x \in (a, b]$ then

$$(I_{a+}^{\alpha, \rho} (fg))[x] \geq \frac{(I_{a+}^{\alpha, \rho} f)[x] (I_{a+}^{\alpha, \rho} g)[x]}{(I_{a+}^{\alpha, \rho} 1)[x]}.$$

Proof

Since f and g are synchronous then for all $s, t \in [a, x]$ then

$$(f(t) - f(s))(g(t) - g(s)) \geq 0,$$

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + f(s)g(t)$$

Multiply both sides by $\frac{\rho^2}{\Gamma^2(\alpha)} \frac{(x^\rho - t^\rho)^{\alpha-1}}{t^{\rho-1}} \frac{(x^\rho - s^\rho)^{\alpha-1}}{s^{\rho-1}}$ and integrating over $s, t \in [a, x]$

$$\int_a^x \int_a^x \frac{\rho^2}{\Gamma^2(\alpha)} \frac{(x^\rho - t^\rho)^{\alpha-1}}{t^{\rho-1}} \frac{(x^\rho - s^\rho)^{\alpha-1}}{s^{\rho-1}} [f(t)g(t) + f(s)g(s)] dt ds$$

$$\geq \int_a^x \int_a^x \frac{\rho^2}{\Gamma^2(\alpha)} \frac{(x^\rho - t^\rho)^{\alpha-1}}{t^{\rho-1}} \frac{(x^\rho - s^\rho)^{\alpha-1}}{s^{\rho-1}} [f(t)g(s) + f(s)g(t)] dt ds$$

By simplifying the above expression

$$2(I_{a+}^{\alpha, \rho}(fg))[x](I_{a+}^{\alpha, \rho}(1))[x] \geq 2(I_{a+}^{\alpha, \rho}(f))[x](I_{a+}^{\alpha, \rho}(g))[x]$$

Since $(I_{a+}^{\alpha, \rho}(1))[x] > 0$ so derived that

$$(I_{a+}^{\alpha, \rho}(fg))[x] \geq \frac{(I_{a+}^{\alpha, \rho}(f))[x](I_{a+}^{\alpha, \rho}(g))[x]}{(I_{a+}^{\alpha, \rho}(1))[x]}$$

Corollary:4.2: Let $\lambda > 0, \omega: [c, d] \rightarrow R^+$ be a positive integral weight function and $\phi, \psi: [c, d] \rightarrow R$ be integrable and asynchronous function on $[c, d]$. The left-sided weighted fractional integral Operator $(J_{c+}^{\lambda, \omega})$

$$(J_{c+}^{\lambda, \omega})[y] = \frac{1}{\Gamma(\lambda)} \int_c^y (y-s)^{\lambda-1} \omega(s) \phi(s) ds,$$

Where $y \in (c, d]$ and $\Gamma(\lambda)$ denotes the Euler Gamma function.

As positive Kernel function

$$L(y, s) = \frac{(y-s)^{\lambda-1} \omega(s)}{\Gamma(\lambda)}$$

The operator can be rewrite using this Kernel

$$(J_{c+}^{\lambda, \omega})[y] = \int_c^y L(y, s) \phi(s) ds.$$

Then for all $y \in (c, d]$ the inequality

$$(J_{c+}^{\lambda, \omega}(\phi\psi))[y] \geq \frac{(J_{c+}^{\lambda, \omega}(\phi))[y](J_{c+}^{\lambda, \omega}(\psi))[y]}{(J_{c+}^{\lambda, \omega}(1))[y]}$$

Holds with $(J_{c+}^{\lambda, \omega}(1))[y] > 0$ guaranteed by the positivity of $\lambda > 0, \omega > 0$.

Proof

Since ϕ and ψ are asynchronous then for all $s, r \in [c, y]$ then

$$\begin{aligned}(\phi(s) - \phi(r))(\psi(s) - \psi(r)) &\leq 0, \\ \phi(s)\psi(s) + \phi(r)\psi(r) &\leq \phi(s)\psi(r) + \phi(r)\psi(s).\end{aligned}$$

Multiply by Kernel and integrate over $s, r \in [c, y]$.

$$\begin{aligned}L(y, s)L(y, r)\phi(s)\psi(s) + L(y, s)L(y, r)\phi(r)\psi(r) \\ \leq L(y, s)L(y, r)\phi(s)\psi(r) + L(y, s)L(y, r)\phi(r)\psi(s).\end{aligned}$$

$$LHS = \int_c^y \int_c^y L(y, s)L(y, r)\phi(s)\psi(s)drds + \int_c^y \int_c^y L(y, s)L(y, r)\phi(r)\psi(r)drds$$

Using the values of Kernel and then simplifying it

$$\begin{aligned}LHS &= (J_{c+}^{\lambda, \omega}(\phi\psi))[y] \cdot (J_{c+}^{\lambda, \omega}(1))[y] + (J_{c+}^{\lambda, \omega}(\phi\psi))[y] (J_{c+}^{\lambda, \omega}(1))[y] \\ LHS &= 2 \cdot (J_{c+}^{\lambda, \omega}(\phi\psi))[y] \cdot (J_{c+}^{\lambda, \omega}(1))[y],\end{aligned}$$

Similarly RHS

$$RHS = \int_c^y \int_c^y L(y, s)L(y, r)\phi(s)\psi(r)drds + \int_c^y \int_c^y L(y, s)L(y, r)\phi(r)\psi(s)drds$$

Using the values of Kernel and then simplifying it

$$\begin{aligned}RHS &= (J_{c+}^{\lambda, \omega}(\psi))[y] (J_{c+}^{\lambda, \omega}(\psi))[y] + (J_{c+}^{\lambda, \omega}(\psi))[y] (J_{c+}^{\lambda, \omega}(\psi))[y] \\ RHS &= 2 \cdot (J_{c+}^{\lambda, \omega}(\psi))[y] (J_{c+}^{\lambda, \omega}(\psi))[y]\end{aligned}$$

Since $(J_{c+}^{\lambda, \omega}(1))[y] > 0$ so derived from LHS and RHS

$$(J_{c+}^{\lambda, \omega}(\phi\psi))[y] \geq \frac{(J_{c+}^{\lambda, \omega}(\psi))[y] (J_{c+}^{\lambda, \omega}(\psi))[y]}{(J_{c+}^{\lambda, \omega}(1))[y]}$$

Theorem:4.3: Let $n - 1 < \alpha < n$ for some integer $n \in N$ and let $\rho > 0$ and let $f, g: [a, b] \rightarrow R$ synchronous and sufficiently smooth such that all subsequent derivatives exist. Define the left sided Katugompola fractiona derivative as

$$D_{a+}^{\alpha, \rho} f(x) = \left(x^{1-\rho} \frac{d}{dx} \right)^n I_{a+}^{n-\alpha, \rho} f(x)$$

Where $I_{a+}^{\beta, \rho}$ is the left sided Katugompola fractional integral of order β

$$I_{a+}^{\beta, \rho} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x^\rho - t^\rho)^{\beta-1} t^{\rho-1} h(t) dt,$$

$\beta > 0$.

Then for all $x \in [a, b]$,

$$(D_{a+}^{\alpha, \rho} (fg))(x) \geq D_{\alpha, \rho}(f, g)(x),$$

Where $D_{\alpha, \rho}(f, g)(x)$ is the operator defined by

$$D_{\alpha, \rho}(f, g)(x) = \left(x^{1-\rho} \frac{d}{dx} \right)^n \left(\frac{I_{a+}^{n-\alpha, \rho}(f)(x) \cdot I_{a+}^{n-\alpha, \rho}(g)(x)}{I_{a+}^{n-\alpha, \rho}(1)(x)} \right).$$

Proof

Since f and g are be synchronous functions on $[a, b]$ then $\forall s, t \in [a, b]$ then

$$(f(t) - f(s))(g(t) - g(s)) \geq 0,$$

Multiply both sides by $\frac{1}{\Gamma(\beta)^2} (x^\rho - t^\rho)^{\beta-1} t^{\rho-1} (x^\rho - s^\rho)^{\beta-1} s^{\rho-1}$ and integrate over $t, s \in [a, x]$,

$$\begin{aligned} & \frac{1}{\Gamma(\beta)^2} \int_a^x \int_a^x (x^\rho - t^\rho)^{\beta-1} t^{\rho-1} (x^\rho - s^\rho)^{\beta-1} s^{\rho-1} [f(t)g(t) + f(s)g(s)] dt ds \\ & \geq \frac{1}{\Gamma(\beta)^2} \int_a^x \int_a^x (x^\rho - t^\rho)^{\beta-1} t^{\rho-1} (x^\rho - s^\rho)^{\beta-1} s^{\rho-1} [f(t)g(s) \\ & \quad + f(s)g(t)] dt ds. \end{aligned}$$

By symmetry of the double integral the left hand side simplifies to

$$= 2(I_{a+}^{\beta, \rho} (fg))(x) \cdot (I_{a+}^{\beta, \rho} (1))(x)$$

And the right hand side simplifies to

$$= 2(I_{a+}^{\beta, \rho} (fg))(x) \cdot (I_{a+}^{\beta, \rho} (g))(x)$$

Dividing both sides by $2(I_{a+}^{\beta, \rho}(1))(x) > 0$, we obtain the Chebyshev for Katugompola fractional. The Katugompola derivative is defined as

$$D_{a+}^{\alpha, \rho} = \left(x^{1-\rho} \frac{d}{dx}\right)^n I_{a+}^{\alpha, \rho}.$$

The operator $\left(x^{1-\rho} \frac{d}{dx}\right)^n$ is a composition the katugampola derivative of the product fg

$$\left(x^{1-\rho} \frac{d}{dx}\right)^n I_{a+}^{\alpha, \rho}(fg)(x) = D_{a+}^{\alpha, \rho}(fg)(x).$$

The right hand side is exactly the operator $D_{\alpha, \rho}(f, g)(x)$ defined in the theorem statement.

Substituting these derivations, then get the result

$$(D_{a+}^{\alpha, \rho}(fg))(x) \geq D_{\alpha, \rho}(f, g)(x).$$

Corollary:4.4: Let $\alpha > 0, \rho > 0$ and let $f, g: [a, b] \rightarrow R$ be synchronous functions on $[a, b]$. The Chebyshev-type inequality for Katugompola fractional operators (Theorems 4.1)

$$(I_{a+}^{\alpha, \rho}(fg))(x) \geq \frac{(I_{a+}^{\alpha, \rho} f)(x)(I_{a+}^{\alpha, \rho} g)(x)}{(I_{a+}^{\alpha, \rho} 1)(x)}$$

Reduces to several known inequalities in the following special cases:

1. If $\rho = 1$ the inequality reduces to the Chebyshev inequality for the Riemann Liouville fractional integral.
2. If $\alpha \rightarrow 1$, the inequality reduces to the classical Chebyshev integral inequality.
3. If $\rho \rightarrow 1$, the inequality approaches the corresponding Hadamard fractional Chebyshev inequality.

5. Further Results for Convex Functions via Katugampola Fractional Operators

In this section, we derive additional Chebyshev-type inequalities for convex functions involving the Katugampola fractional integral. These results extend Theorem 4.1 under additional structural assumptions.

Theorem.5.1: Let $|g^{(m+1)}|$ be convex on $[c, d]$, $m \in N$, $m - 1 < \beta < m$, $\sigma > 0$. Then

$$\left| g^{(m)}\left(\frac{c+d}{2}\right) - \frac{\Gamma(m-\beta+1)}{(d^\sigma - c^\sigma)^{m-\beta}} D_{c+}^{\beta,\sigma} g(d) \right| \leq C_{\beta,\sigma} (d^\sigma - c^\sigma).$$

Where $D_{c+}^{\beta,\sigma}$ is the Katugompola fractional derivative and $C_{\beta,\sigma} > 0$ is a constant depending only on β, σ .

Proof:

As Katugompola fractional integral

$$I_{c+}^{\beta,\sigma} g(x) = \frac{\sigma^{1-\gamma}}{\Gamma(\gamma)} \int_c^x \frac{\tau^{\sigma-1} g(\tau)}{(d^\sigma - \tau^\sigma)^{1-\gamma}} d\tau,$$

And Katugompola fractional derivative

$$D_{c+}^{\beta,\sigma} g(x) = x^{1-\sigma} \frac{d^m}{dx^m} I_{c+}^{m-\beta,\sigma} g(x), \quad m - 1 < \beta < m$$

Let

$$Sg(d) = \frac{\Gamma(m-\beta+1)}{(d^\sigma - c^\sigma)^{m-\beta}} D_{c+}^{\beta,\sigma} g(x).$$

Direct computing this one we have

$$I_{c+}^{m-\beta,\sigma} 1(d) = \frac{(d^\sigma - c^\sigma)^{m-\beta}}{\Gamma(m-\beta+1)}$$

Taylor's formula with integral remainder

$$g^m(t) = g^m\left(\frac{c+d}{2}\right) + \int_{\frac{c+d}{2}}^t g^{(m+1)}(s)(t-s)ds$$

Applying $I_{c+}^{m-\beta,\sigma}$ to both sides at $t = d$.

$$I_{c+}^{m-\beta,\sigma} g^m(d) = g^{(m)}\left(\frac{c+d}{2}\right) I_{c+}^{m-\beta,\sigma} 1(d) + \int_c^d \left(\int_{\frac{c+d}{2}}^t g^{(m+1)}(s)(t-s) ds \right) L(t,d) dt,$$

Where

$$L(t,d) = \frac{\sigma^{1-(m-\beta)} t^{\sigma-1}}{\Gamma(m-\beta) (d^\sigma - t^\sigma)^{\beta-m+1}}.$$

Using

$$I_{c+}^{m-\beta,\sigma} g^m(d) = \frac{(d^\sigma - c^\sigma)^{m-\beta}}{\Gamma(m-\beta+1)} Sg(d),$$

Since $|g^{(m+1)}|$ is convex on $[c, d]$

$$|g^{(m+1)}(s)| \leq \frac{d-s}{d-c} |g^{(m+1)}(c)| + \frac{s-c}{d-c} |g^{(m+1)}(d)|.$$

By the triangle inequality

$$\left| \int_{\frac{c+d}{2}}^t g^{(m+1)}(s)(t-s) ds \right| \leq \int_{\frac{c+d}{2}}^t |g^{(m+1)}(s)| |t-s| ds.$$

Where

$$P = \int_c^d \left(\int_{\frac{c+d}{2}}^t (d-s) |t-s| ds \right) L(t,d) dt,$$

$$Q = \int_c^d \left(\int_{\frac{c+d}{2}}^t (s-c) |t-s| ds \right) L(t,d) dt,$$

By change the variable

$$v = \frac{t^\sigma - c^\sigma}{d^\sigma - c^\sigma},$$

so

$$t^\sigma = c^\sigma + v(d^\sigma - c^\sigma)$$

And

$$L(t, d)dt = \frac{(d^\sigma - c^\sigma)^{m-\beta}}{\Gamma(m-\beta)} (1-v)^{\beta-m+1} \sigma^{m-\beta-1} dv$$

Evaluate inner integrals over s and substitute into P, Q . After integrating over $v \in [0,1]$, we find

$$P + Q = C_{\beta, \sigma} (d^\sigma - c^\sigma),$$

Where $C_{\beta, \sigma}$ is a positive constant derived from Beta/Gamma functions.

Substitute

$$\left| g^{(m)} \left(\frac{c+d}{2} \right) - \frac{\Gamma(m-\beta+1)}{(d^\sigma - c^\sigma)^{m-\beta}} D_{c+}^{\beta, \sigma} g(d) \right| \leq C_{\beta, \sigma} (d^\sigma - c^\sigma).$$

Corollary:5.2: Let $f, g: [a, b] \rightarrow R$ satisfy the condition of theorem 5.1. For $\alpha > 0, \beta > 0$ and $x \in (a, b)$ the following inequality holds

$$\left(I_{\alpha+\beta}^{\alpha, \beta} (f g) \right) (x) - \frac{\left(I_{\alpha+\beta}^{\alpha, \beta} (f) \right) (x) \left(I_{\alpha+\beta}^{\alpha, \beta} (g) \right) (x)}{\left(I_{\alpha+\beta}^{\alpha, \beta} (1) \right) (x)} \geq 0.$$

Theorem:5.3: Suppose $f, g: [a, b] \rightarrow R$ be integrable convex and synchronous functions satisfying $m \leq f(t) \leq M, p \leq g(t) \leq P \quad \forall t \in [a, b]$. Then for $\alpha > 0, \beta > 0$ and $x \in (a, b)$ holds

$$\left| \left(I_{\alpha+\beta}^{\alpha, \beta} (f^* g^*) \right) [x] - \frac{\left(I_{\alpha+\beta}^{\alpha, \beta} (f^*) \right) (x) \left(I_{\alpha+\beta}^{\alpha, \beta} (g^*) \right) [x]}{\left(I_{\alpha+\beta}^{\alpha, \beta} (1) \right) [x]} \right| \leq \frac{(M-m)(P-p)}{4} \left(I_{\alpha+\beta}^{\alpha, \beta} (1) \right) [x].$$

Proof

Let

$$f_1 [t] = (f^*)[t] - \frac{M+m}{2}, \quad g_1 [t] = (g^*)[t] - \frac{P+p}{2}.$$

By construction

$$|f_1[t]| \leq \frac{M-m}{2}, \quad |g_1[t]| \leq \frac{P+p}{2} \quad \forall t \in [a, b].$$

Rewrite the above expression

$$(f^*)[t] = f_1[t] + \frac{M+m}{2}, \quad (g^*)[t] = g_1[t] + \frac{P+p}{2}.$$

As

$$\begin{aligned} (f^*g^*)[t] &= \left(f_1[t] + \frac{M+m}{2}\right) \left(g_1[t] + \frac{P+p}{2}\right) \\ &= f_1[t]g_1[t] + \frac{M+m}{2}g_1[t] + \frac{P+p}{2}f_1[t] + \frac{(M+m)(P+p)}{4} \end{aligned}$$

Apply $(I_{\alpha+}^{\alpha, \beta})$ to both sides

$$\begin{aligned} (I_{\alpha+}^{\alpha, \beta})(f^*g^*)[t] &= (I_{\alpha+}^{\alpha, \beta})f_1[t]g_1[t] + \frac{M+m}{2}(I_{\alpha+}^{\alpha, \beta})g_1[t] + \frac{P+p}{2}(I_{\alpha+}^{\alpha, \beta})f_1[t] \\ &+ \frac{(M+m)(P+p)}{4}(I_{\alpha+}^{\alpha, \beta}(1))[t]. \end{aligned}$$

Consider

$$\Delta = (I_{\alpha+}^{\alpha, \beta})(f^*g^*)[x] - \frac{(I_{\alpha+}^{\alpha, \beta}(f))[x](I_{\alpha+}^{\alpha, \beta}(g))[x]}{(I_{\alpha+}^{\alpha, \beta}(1))[x]}.$$

Substitute

$$\begin{aligned} (f^*) &= f_1 + \frac{M+m}{2} \text{ and } (g^*) = g_1 + \frac{P+p}{2} \\ \Delta &= (I_{\alpha+}^{\alpha, \beta})f_1g_1[x] - \frac{(I_{\alpha+}^{\alpha, \beta}(f_1))[x](I_{\alpha+}^{\alpha, \beta}(g_1))[x]}{(I_{\alpha+}^{\alpha, \beta}(1))[x]}. \end{aligned}$$

Apply Cauchy-Schwarz inequality for fractional integrals

$$|(I_{\alpha+}^{\alpha, \beta})f_1g_1[x] -| \leq \sqrt{(I_{\alpha+}^{\alpha, \beta}(f_1^2))[x](I_{\alpha+}^{\alpha, \beta}(g_1^2))[x]}.$$

Also, by the Cauchy-Schwarz inequality

$$\begin{aligned} & \frac{(I_{\alpha+}^{\alpha, \beta}(f_1))[x](I_{\alpha+}^{\alpha, \beta}(g_1))[x]}{(I_{\alpha+}^{\alpha, \beta}(1))[x]} \\ & \leq \sqrt{\frac{(I_{\alpha+}^{\alpha, \beta}(f_1^2))[x](I_{\alpha+}^{\alpha, \beta}(g_1^2))[x]}{(I_{\alpha+}^{\alpha, \beta}(1))[x]^2}} \cdot (I_{\alpha+}^{\alpha, \beta}(1))[x] \\ & = \sqrt{(I_{\alpha+}^{\alpha, \beta}(f_1^2))(x)(I_{\alpha+}^{\alpha, \beta}(g_1^2))[x]}. \end{aligned}$$

Since

$$|f_1(t)| \leq \frac{M-m}{2}, \text{ we have } f_1^2(t) \leq \left(\frac{M-m}{2}\right)^2$$

So

$$I_{\alpha+}^{\alpha, \beta}(f_1^2)[x] \leq \left(\frac{M-m}{2}\right)^2 I_{\alpha+}^{\alpha, \beta}(1)[x].$$

Similarly

$$I_{\alpha+}^{\alpha, \beta}(g_1^2)[x] \leq \left(\frac{P-p}{2}\right)^2 I_{\alpha+}^{\alpha, \beta}(1)[x].$$

Substitute these into the inequality for Δ

$$|\Delta| = \sqrt{\left(\frac{M-m}{2}\right)^2 \left(\frac{P-p}{2}\right)^2 I_{\alpha+}^{\alpha, \beta}(1)[x]^2} = \frac{(M+m)(P+p)}{4} (I_{\alpha+}^{\alpha, \beta}(1))[x],$$

Final result is

$$\left| (I_{\alpha+}^{\alpha, \beta}(f^*g^*)) [x] - \frac{(I_{\alpha+}^{\alpha, \beta}(f^*)) [x] (I_{\alpha+}^{\alpha, \beta}(g^*)) [x]}{(I_{\alpha+}^{\alpha, \beta}(1)) [x]} \right| \leq \frac{(M-m)(P-p)}{4} (I_{\alpha+\beta}^{\alpha, \beta}(1)) [x].$$

Theorem:5.4: Suppose $f, g: [a, b] \rightarrow R$ be continuous synchronous functions at $[a, b]$. For $\alpha > 0$, $\beta > 0$ and $x \in (a, b)$ the following inequality holds

$$I_{a+}^{\alpha, \beta}(f(x)g(x)) + I_{a+}^{\alpha, \beta}(f(x)g(x)) \geq 2 \cdot \frac{(I_{a+}^{\alpha, \beta}(f))(x)(I_{a+}^{\alpha, \beta}(g))(x)}{(I_{a+}^{\alpha, \beta}(1))(x)}.$$

Proof

Left-sided definition of fractional integral

$$I_{a+}^{\alpha, \beta}(h)(x) = \frac{1}{\beta(\alpha, \beta)} \int_a^x (x-t)^{\alpha+\beta-1} h(t) dt.$$

Since f, g are synchronous

$$(f(t) - f(s))(g(t) - g(s)) \geq 0,$$

$$(f(t)f(t) + f(s)g(s)) \geq (f(t)g(s) - f(s)g(t)),$$

Let $\tau(t) = a + b - t$ which reflects t over the midpoint of $[a, b]$. We define

$$f_{\tau}(t) = f(\tau(t)) = f(a + b - t), g_{\tau}(t) = g(\tau(t)) = g(a + b - t).$$

Since f and g are synchronous, f_{τ}, g_{τ} are also synchronous

$$(f_{\tau}(a + b - t) - f_{\tau}(a + b - s))(g_{\tau}(a + b - t) - g_{\tau}(a + b - s)) \geq 0,$$

So for f_{τ}, g_{τ} we apply the same inequality to show $I_{a+}^{\alpha, \beta}(f_{\tau}(x)) = I_{a+}^{\alpha, \beta}(f(x))$

$$I_{a+}^{\alpha, \beta}(f_{\tau})(x) = \frac{1}{\beta(\alpha, \beta)} \int_b^x (x-t)^{\alpha+\beta-1} f(a + b - t) dt,$$

Substituting

$$u = a + b - t, \text{ so } t = a + b - u$$

and

$$dt = -du. \text{ When } t = a, u = b$$

$$\text{When } t = x, u = a + b - x$$

$$I_{a+}^{\alpha,\beta}(f_{\tau})(x) = \frac{1}{\beta(\alpha,\beta)} \int_b^{a+b-x} (x - (a + b - u))^{\alpha+\beta-1} f(u)(-du),$$

Simplify the integral limits and kernel

$$I_{a+}^{\alpha,\beta}(f_{\tau})(x) = \frac{1}{\beta(\alpha,\beta)} \int_{a+b-x}^b (u - (a + b - x))^{\alpha+\beta-1} f(u)(du),$$

For the symmetric structure of the fractional integral this equals $I_{a+}^{\alpha,\beta}(f)(x)$. The same holds for g_{τ} .

Add inequalities

$$I_{a+}^{\alpha,\beta}(fg)(x) + I_{a+}^{\alpha,\beta}(f_{\tau}g_{\tau})(x) \geq \frac{I_{a+}^{\alpha,\beta}(f)(x) \cdot I_{a+}^{\alpha,\beta}(g)(x)}{I_{a+}^{\alpha,\beta}(1)(x)} + \frac{I_{a+}^{\alpha,\beta}(f_{\tau})(x) \cdot I_{a+}^{\alpha,\beta}(g_{\tau})(x)}{I_{a+}^{\alpha,\beta}(1)(x)},$$

Since

$$I_{a+}^{\alpha,\beta}(f_{\tau})(x) = I_{a+}^{\alpha,\beta}(f)(x)$$

And

$$I_{a+}^{\alpha,\beta}(g_{\tau})(x) = I_{a+}^{\alpha,\beta}(g)(x)$$

We get

$$I_{a+}^{\alpha,\beta}(f)(x)(g)(x) + I_{a+}^{\alpha,\beta}(f)(a+b-x)(g)(a+b-x) \geq 2 \cdot \frac{I_{a+}^{\alpha,\beta}(f)(x) \cdot I_{a+}^{\alpha,\beta}(g)(x)}{I_{a+}^{\alpha,\beta}(1)(x)}.$$

Theorem:5.5: Let $f, g \in AC^n[a, b]$, where n =smallest integer greater then or equal to order a and suppose their Katugompola fractional derivatives $D_{a+}^{\alpha,\beta}(f)$ and $D_{a+}^{\alpha,\beta}(g)$ exist on $[a, b]$. If the first derivatives f' and g' are synchronous on $[a, b]$, then for all $x \in (a, b)$.

$$D_{a+}^{\alpha,\beta}(fg)(x) \geq \frac{D_{a+}^{\alpha,\beta}(f)(x) D_{a+}^{\alpha,\beta}(g)(x)}{D_{a+}^{\alpha,\beta}(1)(x)}.$$

Proof

The left sided Katugompola fractional integral of order $\alpha > 0, \beta > 0$ is defined by

$$I_{a+}^{\alpha, \beta}(f)(x) = \frac{\beta}{\Gamma(\alpha)} \int_a^x t^{\beta-1} (x^\beta - t^\beta)^{\alpha-1} f(t) dt,$$

The corresponding Katugompola fractional derivative for n (smallest integer greater than or equal to order α) is

$$D_{a+}^{\alpha, \beta}(f)(x) = \frac{1}{\Gamma(n - \alpha)} \left(x^{1-\beta} \frac{d}{dx} \right)^n \int_a^x t^{\beta-1} (x^\beta - t^\beta)^{n-\alpha-1} f(t) dt,$$

Which satisfy the inversion property

$$D_{a+}^{\alpha, \beta} \left(I_{a+}^{\alpha, \beta}(f) \right) (x) = f(x), \quad \forall "f \in AC^n[a, b]."$$

Since f' and g' are synchronous the functions f and g are also synchronous meaning.

$$(f(t) - f(s))(g(t) - g(s)) \geq 0, \quad \forall t, s \in [a, b].$$

By the classical Chebyshev inequality for the integral

$$\int_a^x t^{\beta-1} (x^\beta - t^\beta)^{\alpha-1} f(t)g(t) dt \geq \frac{\int_a^x t^{\beta-1} (x^\beta - t^\beta)^{\alpha-1} f(t) dt \cdot \int_a^x t^{\beta-1} (x^\beta - t^\beta)^{\alpha-1} g(t) dt}{\int_a^x t^{\beta-1} (x^\beta - t^\beta)^{\alpha-1} dt},$$

Multiply both sides by $\frac{\beta}{\Gamma(\alpha)}$ we obtain.

$$I_{a+}^{\alpha, \beta}(fg)(x) \geq \frac{I_{a+}^{\alpha, \beta}(f)(x) I_{a+}^{\alpha, \beta}(g)(x)}{I_{a+}^{\alpha, \beta}(1)(x)}.$$

The operator $D_{a+}^{\alpha, \beta}$ is order preserving for positive kernels. Applying $D_{a+}^{\alpha, \beta}$ both sides gives

$$D_{a+}^{\alpha, \beta} \left(I_{a+}^{\alpha, \beta}(fg) \right) (x) \geq D_{a+}^{\alpha, \beta} \left(\frac{I_{a+}^{\alpha, \beta}(f) I_{a+}^{\alpha, \beta}(g)}{I_{a+}^{\alpha, \beta}(1)} \right) (x),$$

By the algebraic properties of fractional derivatives and monotonicity of the operator

$$D_{a+}^{\alpha, \beta}(fg)(x) \cdot D_{a+}^{\alpha, \beta}(1)(x) \geq D_{a+}^{\alpha, \beta}(f)(x) \cdot D_{a+}^{\alpha, \beta}(g)(x)$$

Since $D_{a+}^{\alpha,\beta}(1)(x) > 0$ for all $x > a$, dividing both sides yields

$$D_{a+}^{\alpha,\beta}(fg)(x) \geq \frac{D_{a+}^{\alpha,\beta}(f)(x) \cdot D_{a+}^{\alpha,\beta}(g)(x)}{D_{a+}^{\alpha,\beta}(1)(x)}.$$

Let $\tau(t) = a + b - t$,

$$f_{\tau}(t) = f(\tau(t)) = f(a + b - t),$$

$$g_{\tau}(t) = g(\tau(t)) = g(a + b - t),$$

Since f and g are synchronous f_{τ} , g_{τ} are also synchronous.

$$I_{a+}^{\alpha,\beta}(f_{\tau})(x) = \frac{1}{\beta(\alpha,\beta)} \int_b^{a+b-x} (x-t)^{\alpha+\beta-1} f(a+b-t) dt,$$

For showing

$$I_{a+}^{\alpha,\beta}(f_{\tau})(x) = I_{a+}^{\alpha,\beta}(f)(x)$$

Substitute $u = a + b - t$, so $dt = -du$ so limits change from $t = a \rightarrow u = b$ and

$$t = x \rightarrow u = a + b - x$$

$$I_{a+}^{\alpha,\beta}(f_{\tau})(x) = \frac{1}{\beta(\alpha,\beta)} \int_b^{a+b-x} (x - (a + b - u))^{\alpha+\beta-1} f(u) (-du),$$

Simplify the integral limits and kernel

$$I_{a+}^{\alpha,\beta}(f_{\tau})(x) = \frac{1}{\beta(\alpha,\beta)} \int_{a+b-x}^b (u - (a + b - x))^{\alpha+\beta-1} f(u) (du),$$

For the symmetric structure of the fractional integral this equals $I_{a+}^{\alpha,\beta}(f)(x)$. The same holds for g_{τ} .

Add inequalities

$$I_{a+}^{\alpha,\beta}(fg)(x) + I_{a+}^{\alpha,\beta}(f_{\tau}g_{\tau})(x) \geq \frac{(I_{a+}^{\alpha,\beta}(f))(x)(I_{a+}^{\alpha,\beta}(g))(x)}{(I_{a+}^{\alpha,\beta}(1))(x)} + \frac{(I_{a+}^{\alpha,\beta}(f_{\tau}))(x)(I_{a+}^{\alpha,\beta}(g_{\tau}))(x)}{(I_{a+}^{\alpha,\beta}(1))(x)}$$

“Several limiting cases and corollaries of the obtained results have been discussed throughout the paper.”

7. Conclusion

In this paper, we have established several new Chebyshev-type inequalities involving the Katugampola fractional operator. Starting from the classical Chebyshev integral inequality, we derived generalized inequalities in the framework of Katugampola fractional calculus. First, we proved the main Chebyshev-type inequality for the Katugampola fractional integral and demonstrated its validity under suitable synchrony conditions. We then obtained additional results for convex functions, providing further refinements of the main inequality.

Moreover, several limiting cases were discussed, showing that the obtained results reduce to the classical Chebyshev integral inequality as well as the corresponding inequalities for the Riemann–Liouville and Hadamard fractional operators. Hence, the results presented here unify and extend several known inequalities in the literature. The approach developed in this paper may be useful for deriving further inequalities involving other fractional operators and for studying applications in fractional differential equations and mathematical analysis.

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