Transmuted Exponential-Weibull-Exponential Distribution and its Parameter Effects on Median, Mean, Variance, Skewness and Kurtosis: Application to Lifetime Data

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Abstract- This study presents a new lifetime model known as the Transmuted Exponential-Weibull-Exponential (TE-W-E) distribution. The proposed model possesses a flexible structure that makes it well suited for analyzing positive data and accommodating a bathtub-shaped hazard rate function, which is common in survival analysis. Several fundamental mathematical properties of the new distribution are derived and presented in closed form. These include the ordinary moments, mean, variance, moment generating, quantile, survival, hazard, reversed hazard, odd, cumulative hazard functions, Rényi entropy, and order statistics. The parameter effects on descriptive statistics are examined and the findings show that they have effect on the distribution. The parameters of the proposed TE-W-E model are estimated using the maximum likelihood estimation method. A simulation study is conducted to evaluate the performance of the maximum likelihood estimates in terms of bias, variance, and mean squared error across various sample sizes. The simulation results indicate that the maximum likelihood estimation method provides reliable estimates of the model parameters. To demonstrate the practical usefulness of the proposed distribution, it is fitted to two real lifetime datasets. The results show that the proposed TE-W-E model provides a superior goodness-of-fit compared to existing models, based on standard statistical evaluation criteria.

Keywords:Maximum likelihood, Transmuted Exponential-Weibull-Exponential, Weibull-Exponential, Weibull-Gamma, lifetime data

I. INTRODUCTION

Life data analysis, also known as survival analysis, is a statistical approach used to analyze data on the failure time of a unit or system. The interest in survival analysis is to fit a statistical distribution to lifetime data, that is, failure time data that are collected from a sample of units or systems and to use the fitted distribution to make predictions about the lives of units or systems in the population (Leemis, 1995; ReliaSoft, 2015; Eze & Yahya, 2025a; 2025b). Lifetime data is a term that refers to measurements of a unit's or system's lifespan. It can be obtained from an individual suffering a particular disease, such as data on cancer, gonorrhea, syphilis, human immune-deficiency virus, tuberculosis, hepatitis-B virus, chlamydia, malaria, typhoid and so on (Eze et al., 2022). It can also be obtained from non-living objects such as data on lifetime of battery cells, car engine, electric bulb, components in electronics sets, and so on. The unit or system's lifetime can be measured in minutes, hours, years, cycles-to-failure, miles, stress cycles, or any other metric that can be used to measure the life or exposure of a unit or system. The most common measure of life is time; therefore, lifetime data points are often called times-to-failure data (Weibull.com, 2021). From a survival analysis perspective, there is a mean life for each unit or system in the population which is anticipated to operate under a specified condition. Moreover, survival analysis is indispensable since the knowledge of it helps to reduce hazard rate, strengthen management, and most importantly give much attention to survival of units or systems (ReliaSoft, 2015).

Lifetime data are well described using lifetime distributions, but not all lifetime distributions are appropriate for modeling some special lifetime data. Instead of transposing the lifetime data, it is better to extend the existing models, that is, to use statistical or mathematical approach to develop a new model that will best fit the data. Some of the most widely applicable lifetime distributions includes Weibull, exponential, gamma, normal, lognormal, and Lindley distributions. Many authors introduced different techniques to extend or generate statistical distributions for modeling data for a better fit. The formation of a new distribution from the existing one is mostly done by inducing additional parameters to the baseline model.

Some famous methods of developing new distributions include; Marshall-Olkin-G by Cordeiro et al. (2014), Beta-G (B-G) by Eugene et al. (2002), Beta Weibull-G by Yousof et al. (2011), Kumaraswamy Transmuted-G by Afify et al. (2016), Gompertz-G by Alizadeh et al. (2017), Transmuted-G by Shaw and Buckley (2007), Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011), Exp-G (EG) class by Cordeiro et al. (2013), Lom-G class by Cordeiro et al. (2014), Weibull-G by Bourguignon et al. (2014), Transmuted Exponentiated Generalized-G (TExG-G) by Yousof et al. (2015), Transmuted Exponential-G by Mohammed and Ugwuowo (2020). In literature, numerous families of Transmuted-G generators by Shaw and Buckley (2007) have been studied, and proven that these distributions give better fits than the baseline distributions. Some of these distributions include; Transmuted Rayleigh by Merovci (2013c), Transmuted Lindley by Merovci (2013a), Transmuted exponential by Shaw and Buckley (2009), Transmuted exponential exponential by Merovci (2013b), Transmuted Gumbel by Aryal and Tsokos (2009), Transmuted Fréchet by Mahmoud and Mandouh (2013), Transmuted log-logistic by Aryal (2013), Transmuted Lomax by Ashour and Eltehiwy (2013), Transmuted Kumaraswamy by Ahmad et al. (2015), Transmuted Weibull by Aryal and Tsokos (2011), Transmuted modified inverse Weibull by Elbatal (2013b), Transmuted exponentiated gamma by Hussian (2014), Transmuted Gompertz by Abdul-Moniem and Seham (2015), Transmuted modified Weibull by Khan and King (2013), Transmuted complementary Weibull Geometric by Afify et al. (2014), Transmuted Weibull Lomax by Afify et al. (2015), Transmuted Exponentiated Weibull Geometric by Saboor et al. (2016), Transmuted New Weibull-Pareto by Tahir et al. (2018), Transmuted Exponential-Weibull by Mohammed and Ugwuowo(2020), Transmuted Topp-Leone Weibull by Ibrahim and Yousof (2014) etc.

This study focuses on extending Weibull-Exponential (W-E) distribution and deriving its properties such as mean, variance, moment generating function, entropy and order statistics. The W-E distribution is an extension of Weibull distribution and it was obtained from the modification of Weibull-Gamma (W-G) distribution as cited by Qutb and Rajhi (2016). It was developed in order to increase the flexibility of Weibull distribution (Weibull, 1949; 1951). Therefore, the researchers in the current study were motivated to increase the flexibility of the W-E distribution by including a new parameter using Transmuted Exponential-G (TE-G) generator of distribution technique that was proposed by Mohammed and Ugwuowo (2020). The proposed lifetime distribution in this study is called Transmuted Exponential-Weibull-Exponential (TE-W-E) distribution. The proposed TE-W-E distribution is to be used to perform survival analysis. The common functions in survival analysis are derived from the proposed TE-W-E distribution which include; the survival function, hazard or failure rate function, reversed hazard function, odd function and cumulative hazard function. The proposed TE-W-E distribution is highly flexible and has the ability to capture all the information at the extreme of a given asymmetric dataset that skewed to the right, therefore, solving the problem of choosing a lifetime distribution that best fits the datasets that are characterized with elongation and asymmetry. Its practical significance arises from its ability to model different types of hazard functions, which makes it suitable across many disciplines such as medicine, engineering, psychology, pharmacy, veterinary medicine, meteorology, environmental sciences etc.

Further interest in this study is to use the proposed TE-W-E distribution to model lifetime data in medicine. In medicine, modeling the lifetime of a patient suffering a certain disease until events such as death, disease remission, or relapse occurs is a common practice. The TE-W-E distribution adapts to increasing or decreasing hazard over time, representing the biological realities of disease progression or treatment efficacy. The real-world medical contexts or scenarios in which the TE-W-E model would offer advantages or improved performance over existing distributions are (a) modeling time to death or relapse of cancer case because of its ability to increasing or decreasing hazard over time, (b) modeling time to heart attack of cardiovascular risk because of its ability to capture age-related rising hazard, (c) modeling time to readmission of a patient since it can capture early post-discharge risk, (d) modeling time to treatment failure of drug effectiveness because of its ability to capture waning effect, (e) modeling time to medical device failure due to its ability to model early or wear-out failures, (f) modeling time to dropout or adverse event during clinical trial participation because of its ability to supports realistic trial planning, (g) modeling latency to symptoms or death of infectious disease progression because of its ability to capture evolving decrease risk, and (h) modeling time to mortality of geriatric care because it has the ability to capture increasing mortality risk and so on. Furthermore, in healthcare facilities, the TE-W-E model performs efficiently well in modeling non-constant hazard rates that are common in patient survival, treatment response, medical device failure, and disease progression. In addition, the TE-W-E model offers better interpretability, predictive power, and model fit in scenarios involving life expectancy, treatment dynamics, and time-dependent medical events when compared to exponential, normal, or gamma distributions.

The remaining parts of the article are outlined in sections as follows. In Section 2, we defined the TE-W-E distribution and show that it is a valid pdf. In section 3, we derived and presented some general mathematical and statistical properties of the proposed TE-W-E distribution which include; ordinary moments, moment generating function, mean, variance, coefficient of skewness, coefficient of kurtosis, survival function, hazard function, reversed hazard function, odd function and cumulative hazard function, quantile function, Renyi entropy and order statistics. In section 4, maximum likelihood estimation (MLE) of the model parameters is investigated. In section 5, simulation study is carried out to evaluate the behavior of median, mean, variance, skewness and kurtosis of TE-W-E model. In section 6, simulation study is carried out to evaluate the performance of maximum likelihood estimators. In section 7, we illustrated the potentiality of the proposed TE-W-E distribution using real dataset. In section 8, the article was concluded.

2. METHODS

In this section, we defined the TE-W-E distribution, show that it is a valid pdf, and prove some of its properties which include; ordinary moments, moment generating function, mean, variance, quantile function, survival function, hazard function, reversed hazard function, odd function, cumulative hazard function, Renyi entropy and order statistics.

The W-E distribution is a flexible model which was obtained from the Weibull-Gamma (W-G) distribution that was proposed by Dubey (1968). The W-E is a special case of W-G and it was obtained by using a simple transformation on the one of its shape parameters (Qutb, 2016).

Let X be a random variable that distributed as Weibull-Exponential (W-E) distribution; then, its cumulative distribution function (cdf) is given as

$$F(x;\alpha,\delta)=1-\frac{\delta}{(x^{\alpha}+\delta)} \quad (1)$$

The corresponding probability density function (pdf) of W-E is given as

$$f(x;\alpha,\delta)=\frac{\alpha\delta x^{\alpha-1}}{(x^{\alpha}\ +\ \delta)^{2}} \quad (2)$$

where α is the shape parameter andδ is a scale parameter with $x>0$, $\alpha>0$, and $\delta>0$(Dubey, 1968).

2.0 Transmuted Exponential-Weibull-Exponential (TE-W-E) Model

2.1 Definition 1

According to Mohammed and Ugwuowo (2020), a random variable X is said to have a Transmuted Exponential-G (TE-G) distribution function if its cdf and pdf are respectively given as:

$$F(x;\tau,\omega,\xi)=\left(1-\left(1-G(x;\xi)\right)^{\tau}\right)\left(1+\omega\left(1-G(x;\xi)\right)^{\tau}\right) \quad (3)$$

and its corresponding probability density function (pdf) is defined as

$$f(x;\tau,\omega,\xi)=\frac{\tau\left(g(x;\xi)\right)}{1-G(x;\xi)}\left(1-G(x;\xi)\right)^{\tau}\left(1-\omega + 2\omega\left(1-G(x;\xi)\right)^{\tau}\right) \quad (4)$$

where $G(x;\xi)$ is the cdf of any continuous distribution which depends on a parameter vector ξ, $g(x;\xi)$ is its corresponding pdf which depends on a parameter vector ξ, τ is additional scale parameter; $\tau>0$ and ω is the transmuted parameter; $-1\le\omega\le1$

Substituting Eq. (1) and Eq. (2) in Eq. (3) and Eq. (4) and simplifying, we obtain the cdf and pdf of TE-W-E distribution as follows:

$$F(x;\tau,\omega,\alpha,\delta)=\left(1-\left(\frac{\delta}{(x^{\alpha}+\delta)}\right)^{\tau}\right)\left(1+\omega\left(\frac{\delta}{(x^{\alpha}+\delta)}\right)^{\tau}\right) \quad (5)$$

and substituting Eq. (1) and Eq. (2) in Eq. (4) and simplifying, we obtain the pdf of the proposedTE-W-E distribution as:

$$f(x;\tau,\omega,\alpha,\delta)=\frac{\tau\alpha\delta^{\tau}x^{\alpha-1}}{(x^{\alpha}+\delta)^{\tau+1}}(1-\omega)+\frac{2\omega\tau\alpha\delta^{2\tau}x^{\alpha-1}}{(x^{\alpha}+\delta)^{2\tau+1}} \quad (6)$$

where α is the shape parameters, δ is a scale parameter while τ is additional scale parameter from TE-G family and ω is the transmuted parameter; $x>0, \tau>0, \alpha>0,\ \delta>0$ and $|\omega|\le1$

2.2 Validity of the model

The TE-W-E distribution is said to be valid pdf if the following integral in Eq. (7) holds

$$\int_0^{\infty}f(x;\tau,\omega,\alpha,\delta)dx=1 \quad (7)$$

$$\int_0^{\infty}f(x;\tau,\omega,\alpha,\delta)dx=\int_0^{\infty}\left(\frac{\tau\alpha\delta^{\tau}x^{\alpha-1}}{(x^{\alpha}+\delta)^{\tau+1}}(1-\omega)+\frac{2\omega\tau\alpha\delta^{2\tau}x^{\alpha-1}}{(x^{\alpha}+\delta)^{2\tau+1}}\right)dx$$

$$=\int_0^{\infty}\alpha x^{\alpha-1}\left(\frac{\tau\delta^{\tau}}{\left(\delta\left[\left(\frac{x}{\delta^{1/\alpha}}\right)^{\alpha}+1\right]\right)^{\tau+1}}(1-\omega)+\frac{2\omega\tau\delta^{2\tau}}{\left(\delta\left[\left(\frac{x}{\delta^{1/\alpha}}\right)^{\alpha}+1\right]\right)^{2\tau+1}}\right)dx \quad (8)$$

Let

$$p=\left(\frac{x}{\delta^{1/\alpha}}\right)^{\alpha}$$

$$dp/_{dx}=\frac{\alpha x^{\alpha-1}}{\delta}\Rightarrow dx=\frac{\delta dp}{\alpha x^{\alpha-1}}$$

Substituting p and dx in Eq. (8), we have

$$\int_0^{\infty}\alpha x^{\alpha-1}\left(\frac{\tau\delta^{\tau}}{(\delta[p+1])^{\tau+1}}(1-\omega)+\frac{2\omega\tau\delta^{2\tau}}{(\delta[p+1])^{2\tau+1}}\right)\frac{\delta dp}{\alpha x^{\alpha-1}}$$

$$=\int_0^{\infty}\left(\frac{\tau\delta^{\tau+1}(p+1)^{-(\tau+1)}}{\delta^{\tau+1}}(1-\omega)+\frac{2\omega\tau\delta^{2\tau+1}(p+1)^{-(2\tau+1)}}{\delta^{2\tau+1}}\right)dp$$

$$=\frac{\tau(1-\omega)}{(-(\tau+1)+1)}\left((p+1)^{-(\tau+1)+1}\right)_0^{\infty}+\frac{2\omega\tau}{(-(2\tau+1)+1)}\left((p+1)^{-(2\tau+1)+1}\right)_0^{\infty}$$

$$=-\left(\frac{(1-\omega)}{(p+1)^{\tau}}\right)_0^{\infty}-\left(\frac{\omega}{(p+1)^{2\tau}}\right)_0^{\infty}$$

$$=-\left[\frac{(1-\omega)}{(\infty+1)^{\tau}}-\frac{(1-\omega)}{(0+1)^{\tau}}\right]-\left[\frac{\omega}{(\infty+1)^{2\tau}}-\frac{\omega}{(0+1)^{2\tau}}\right] \quad (9)$$

$$=-[0-(1-\omega)]-[0-\omega]=1 \quad (10)$$

2.3 Graphical Presentation of pdf and cdf of TE-W-E Distribution

The plots of the pdf and cdf of the TE-W-E distribution are respectively display in Figure 1 and Figure 2 using some selected values.

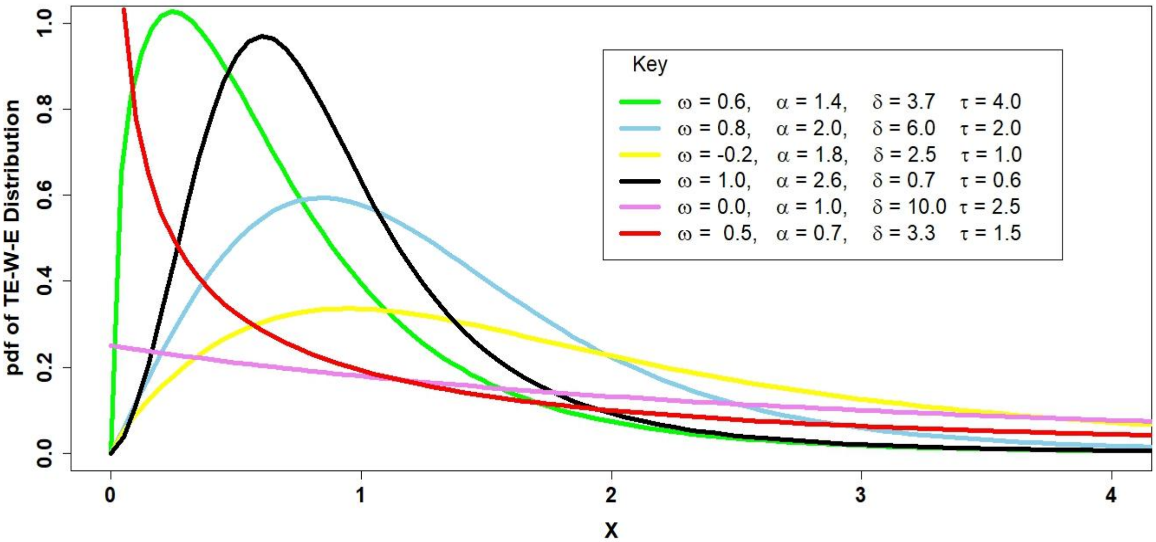

Figure 1: Plot of pdf for some arbitrary parameter values of TE-W-E distribution

Figure 1shows that the pdf of the proposed TE-W-E distribution is positively skewed and thus, will be a good modelfor modeling positive skewed datasets. The density also showed different degrees of skewness and kurtosis due to changes in the shape parameter, α values.

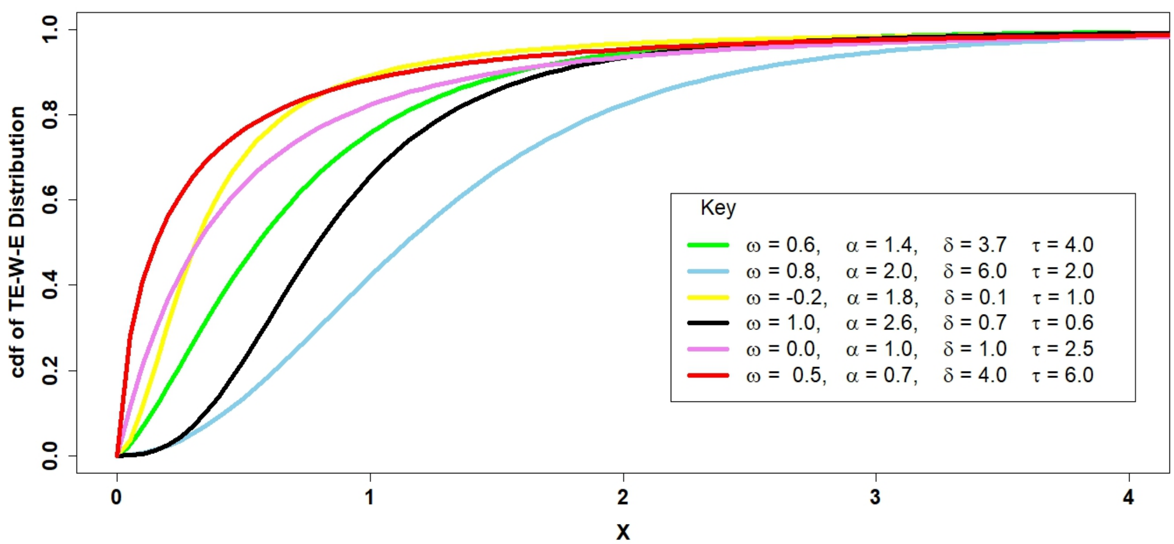

Figure 2: Plot of cdf for some arbitrary parameter values of TE-W-E distribution

Figure 2 shows that the cdf of the proposed TE-W-E distribution is a non-decreasing function. It shows that the line in the cdf plot of the TE-W-E distribution starts from zero and trends upward. It also shows that there is a linear relationship between time and unreliability function, $F(x)$.This will makethe monitoring of the unreliability of units to start from the time zero.

3. MATHEMATICAL PROPERTIES OF TE-W-E DISTRIBUTION

In this section, some properties of the TE-W-E distribution were presented and discussed.

3.1 Moments of TE-W-E Distribution

Suppose a random variable X follows TE-W-E distribution with parameters $\tau, \omega, \alpha,$ andδ, as it was defined in Eq. (6); then the sth moments of TE-W-E distribution is obtained by

$$\mu_s' = E(x^s) = \int_0^\infty x^s f(x; \tau, \omega, \alpha, \delta) dx$$

$$= \int_0^\infty x^s \left(\frac{\tau\alpha\delta^\tau x^{\alpha-1}}{(x^\alpha+\delta)^{\tau+1}}(1-\omega) + \frac{2\omega\tau\alpha\delta^{2\tau}x^{\alpha-1}}{(x^\alpha+\delta)^{2\tau+1}} \right) dx$$

$$= \int_0^\infty x^s \alpha x^{\alpha-1} \left(\frac{\tau\delta^\tau}{\left(\delta\left[\left(\frac{x}{\delta^{1/\alpha}}\right)^\alpha+1\right]\right)^{\tau+1}}(1-\omega) + \frac{2\omega\tau\delta^{2\tau}}{\left(\delta\left[\left(\frac{x}{\delta^{1/\alpha}}\right)^\alpha+1\right]\right)^{2\tau+1}} \right) dx \quad (11)$$

Let

$$p = \left(\frac{x}{\delta^{1/\alpha}}\right)^\alpha \Rightarrow x = p^{1/\alpha}\delta^{1/\alpha}$$

$$dp/_{dx} = \frac{\alpha x^{\alpha-1}}{\delta} \Rightarrow dx = \frac{\delta dp}{\alpha x^{\alpha-1}}$$

Substituting p, x, and dx in Eq. (11), we have

$$\mu_s' = \int_0^\infty \left(p^{1/\alpha}\delta^{1/\alpha}\right)^s \alpha x^{\alpha-1} \left(\frac{\tau\delta^\tau}{(\delta[p+1])^{\tau+1}}(1-\omega) + \frac{2\omega\tau\delta^{2\tau}}{(\delta[p+1])^{2\tau+1}} \right) \frac{\delta dp}{\alpha x^{\alpha-1}} \quad (12)$$

$$= \int_0^\infty \left(p^{1/\alpha}\delta^{1/\alpha}\right)^s \left(\frac{\tau(1-\omega)}{(p+1)^{\tau+1}} + \frac{2\omega\tau}{(p+1)^{2\tau+1}} \right) dp \quad (13)$$

$$= \delta^{s/\alpha} \left(\tau(1-\omega)\int_0^\infty \frac{p^{s/\alpha}}{(p+1)^{\tau+1}} dp + 2\omega\tau \int_0^\infty \frac{p^{s/\alpha}}{(p+1)^{2\tau+1}} dp \right) \quad (14)$$

Recall that

$$\int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}} dx = B(m,n) \quad (15)$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

The Eq. (15) is a standard integral identity involving the Beta function, denoted as $B(m,n)$, and it holds for $m > 0$ and $n > 0$.

Let

$$h = m - 1 \Rightarrow m = h + 1 \quad (16)$$

and $\quad b = m + n \Rightarrow n = b - m \quad (17)$

Substituting Eq. (16) in Eq. (17), we have

$$n = b - h - 1 \quad (18)$$

Substituting Eq.s (16) and Eq. (18) in Eq. (15), we have

$$\int_0^\infty \frac{x^h}{(x+1)^b} dx = B(h+1, b-h-1) \quad (19)$$

$$= \frac{\Gamma(h+1)\Gamma(b-h-1)}{\Gamma(b)} \quad (20)$$

The entire workout show that from Eq. (14)

$$\int_0^\infty \frac{p^{s/\alpha}}{(p+1)^{\tau+1}} dv = \frac{\Gamma\left(\frac{s}{\alpha}+1\right)\Gamma\left(\tau-\frac{s}{\alpha}\right)}{\Gamma(\tau+1)} \quad (21)$$

$$\int_0^\infty \frac{p^{s/\alpha}}{(p+1)^{2\tau+1}} dv = \frac{\Gamma\left(\frac{s}{\alpha}+1\right)\Gamma\left(2\tau-\frac{s}{\alpha}\right)}{\Gamma(2\tau+1)} \quad (22)$$

Substituting Eq.s (21) and Eq. (22) in Eq. (14), we have

$$\mu_s' = \delta^{s/\alpha} \left(\frac{\tau(1-\omega)\Gamma\left(\frac{s}{\alpha}+1\right)\Gamma\left(\tau-\frac{s}{\alpha}\right)}{\Gamma(\tau+1)} + \frac{2\omega\tau\Gamma\left(\frac{s}{\alpha}+1\right)\Gamma\left(2\tau-\frac{s}{\alpha}\right)}{\Gamma(2\tau+1)} \right)$$

$$= \tau\delta^{s/\alpha} \left((1-\omega)B\left(\frac{s}{\alpha}+1, \tau-\frac{s}{\alpha}\right) + 2\omega B\left(\frac{s}{\alpha}+1, 2\tau-\frac{s}{\alpha}\right) \right) \quad (23)$$

where $\tau, \alpha > s$ and s is finite.

3.1.1 *sth* Moments for $s = 1$ and 2

Using Eq.s (23), the corresponding moments for $s = 1$ and 2 are as follows

If $s = 1$

$$\mu_1' = \tau\delta^{1/\alpha} \left((1-\omega)B\left(\tau-\frac{1}{\alpha}, \frac{1}{\alpha}+1\right) + 2\omega B\left(2\tau-\frac{1}{\alpha}, \frac{1}{\alpha}+1\right) \right)$$

If $s = 2$

$$\mu_2' = \tau\delta^{2/\alpha} \left((1-\omega)B\left(\tau-\frac{2}{\alpha}, \frac{2}{\alpha}+1\right) + 2\omega B\left(2\tau-\frac{2}{\alpha}, \frac{2}{\alpha}+1\right) \right)$$

3.2 Mean and Variance of TE-W-E Distribution

3.2.1 Mean of TE-W-E

The first moment, that is when $s = 1$ is the mean of TE-W-E distribution and defined as

$$\mu_1' = \tau\delta^{1/\alpha} \left(\frac{(1-\omega)\Gamma\left(\tau-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma(\tau+1)} + \frac{2\omega\Gamma\left(2\tau-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma(2\tau+1)} \right)$$

$$= \tau\delta^{1/\alpha} \left((1-\omega)B\left(\tau-\frac{1}{\alpha}, \frac{1}{\alpha}+1\right) + 2\omega B\left(2\tau-\frac{1}{\alpha}, \frac{1}{\alpha}+1\right) \right)$$

3.2.2 Variance of TE-W-E

$$V_x = \mu_2' - (\mu_1')^2$$

$$V_x = \tau\delta^{2/\alpha} \left(\frac{(1-\omega)\Gamma\left(\tau-\frac{2}{\alpha}\right)\Gamma\left(\frac{2}{\alpha}+1\right)}{\Gamma(\tau+1)} + \frac{2\omega\Gamma\left(2\tau-\frac{2}{\alpha}\right)\Gamma\left(\frac{2}{\alpha}+1\right)}{\Gamma(2\tau+1)} \right) - \left\{ \tau\delta^{1/\alpha} \left(\frac{(1-\omega)\Gamma\left(\tau-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma(\tau+1)} + \frac{2\omega\Gamma\left(2\tau-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma(2\tau+1)} \right) \right\}^2$$

$$= \tau\delta^{2/\alpha} \left((1-\omega)B\left(\tau-\frac{2}{\alpha}, \frac{2}{\alpha}+1\right) + 2\omega B\left(2\tau-\frac{2}{\alpha}, \frac{2}{\alpha}+1\right) \right) - \left\{ \tau\delta^{1/\alpha} \left((1-\omega)B\left(\tau-\frac{1}{\alpha}, \frac{1}{\alpha}+1\right) + 2\omega B\left(2\tau-\frac{1}{\alpha}, \frac{1}{\alpha}+1\right) \right) \right\}^2$$

3. 3 *Sth* Moment about the Mean for TE-W-E Distribution

The *sth* moment about the mean for TE-W-E distribution is obtained by

$$E(x-\mu)^s = \int_0^\infty (x-\mu)^s f(x;\tau,\omega,\alpha,\delta) dx \quad (24)$$

Substituting Eq. (6) in Eq. (24), we have

$$E(x-\mu)^s = \int_0^\infty (x-\mu)^s \left(\frac{\tau\alpha\delta^\tau x^{\alpha-1}}{(x^\alpha+\delta)^{\tau+1}}(1-\omega) + 2\omega\frac{\tau\alpha\delta^{2\tau}x^{\alpha-1}}{(x^\alpha+\delta)^{2\tau+1}}\right)dx$$

By applying Binomial expansion

$$(x-\mu)^s = \sum_{p=0}^{s}(-1)^p\binom{s}{p}x^{s-p}\mu^p$$

$$E(x-\mu)^s = \int_0^\infty \sum_{p=0}^{s}(-1)^p\binom{s}{p}x^{s-p}\mu^p\alpha x^{\alpha-1}\left(\frac{\tau\delta^\tau}{\left(\delta\left[\left(\frac{x}{\delta^{1/\alpha}}\right)^\alpha+1\right]\right)^{\tau+1}}(1-\omega) + 2\omega\frac{\tau\delta^{2\tau}}{\left(\delta\left[\left(\frac{x}{\delta^{1/\alpha}}\right)^\alpha+1\right]\right)^{2\tau+1}}\right)dx \quad (25)$$

Let

$$v = \left(\frac{x}{\delta^{1/\alpha}}\right)^\alpha \Rightarrow x = v^{1/\alpha}\delta^{1/\alpha}$$

$$dv/_{dx} = \frac{\alpha x^{\alpha-1}}{\delta} \Rightarrow dx = \frac{\delta dv}{\beta x^{\beta-1}}$$

Substitutingv, x, and dx in Eq. (25), we have

$$E(x-\mu)^s = \int_0^\infty \sum_{p=0}^{s}(-1)^p\binom{s}{p}\left(v^{1/\alpha}\delta^{1/\alpha}\right)^{s-p}\mu^p\left(\frac{\tau}{([v+1])^{\tau+1}}(1-\omega) + \frac{2\omega\tau}{([v+1])^{2\tau+1}}\right)dv$$

$$= \sum_{p=0}^{s}(-1)^p\binom{s}{p}\mu^p\left((1-\omega)\int_0^\infty \frac{v^{\frac{s-p}{\alpha}}}{(v+1)^{\tau+1}}dv + 2\omega\int_0^\infty \frac{v^{\frac{s-p}{\alpha}}}{(v+1)^{2\tau+1}}dv\right)\tau\delta^{\left(\frac{s-p}{\alpha}\right)} \quad (26)$$

Recall that

$$\int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}}dx = B(m,n) \quad (27)$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

The Eq. (27) is a standard integral identity involving the Beta function, denoted as $B(m,n)$, and it holds for $m>0$ and $n>0$.

Let

$$h = m-1 \Rightarrow m = h+1 \quad (28)$$

and

$$b = m+n \Rightarrow n = b-m \quad (29)$$

Substituting Eq. (27) in Eq. (28), we have

$$\Rightarrow n = b-h-1 \quad (30)$$

Substituting Eq.s (28) and Eq. (30) in Eq. (27), we have

$$\int_0^\infty \frac{x^h}{(x+1)^b}dx = B(h+1, b-h-1) \quad (31)$$

$$= \frac{\Gamma(h+1)\Gamma(b-h-1)}{\Gamma(b)} \quad (32)$$

The entire workout show that from Eq. (26)

$$\int_0^\infty \frac{v^{s/\alpha}}{(v+1)^{\tau+1}}dv = \frac{\Gamma\left(\frac{s}{\alpha}+1\right)\Gamma\left(\tau-\frac{s}{\alpha}\right)}{\Gamma(\tau+1)} \quad (33)$$

$$\int_0^\infty \frac{v^{s/\alpha}}{(v+1)^{2\tau+1}}dv = \frac{\Gamma\left(\frac{s}{\alpha}+1\right)\Gamma\left(2\tau-\frac{s}{\alpha}\right)}{\Gamma(2\tau+1)} \quad (34)$$

Substituting Eq.s (33) and Eq. (34) in Eq. (26), we have

$$E(x-\mu)^s = \sum_{p=0}^{s}(-1)^p\binom{s}{p}\mu^p\left(\frac{(1-\omega)\Gamma\left(\frac{s-p}{\alpha}+1\right)\Gamma\left(\tau-\frac{s-p}{\alpha}\right)}{\Gamma(\tau+1)} + \frac{2\omega\Gamma\left(\frac{s-p}{\alpha}+1\right)\Gamma\left(2\tau-\frac{s-p}{\alpha}\right)}{\Gamma(2\tau+1)}\right)\tau\delta^{\left(\frac{s-p}{\alpha}\right)}$$

$$= \sum_{p=0}^{s}(-1)^p\binom{s}{p}\mu^p\left((1-\omega)B\left(\left(\frac{s-p}{\alpha}\right)+1, \tau-\left(\frac{s-p}{\alpha}\right)\right) + 2\omega B\left(\left(\frac{s-p}{\alpha}\right)+1, 2\tau-\left(\frac{s-p}{\alpha}\right)\right)\right)\tau\delta^{\left(\frac{s-p}{\alpha}\right)}$$

3.3.1 *Sth* Moment about the Mean for $s = 1$ and 2

If s = 1

$$E(x-\mu) = \sum_{p=0}^{s}(-1)^p\binom{1}{p}\mu^p\left((1-\omega)B\left(\tau-\tfrac{1-p}{\alpha}, \tfrac{1-p}{\alpha}+1\right) + 2\omega B\left(2\tau-\tfrac{1-p}{\alpha}, \tfrac{1-p}{\alpha}+1\right)\right)\tau\delta^{\frac{1-p}{\alpha}}$$

If $s = 2$

$$E(x-\mu)^2 = \sum_{p=0}^{s}(-1)^p\binom{2}{p}\mu^p\left((1-\omega)B\left(\tau-\tfrac{2-p}{\alpha}, \tfrac{2-p}{\alpha}+1\right) + 2\omega B\left(2\tau-\tfrac{2-p}{\alpha}, \tfrac{2-p}{\alpha}+1\right)\right)\tau\delta^{\frac{2-p}{\alpha}}$$

3.4 Coefficient of Variation of TE-W-E Distribution

The coefficient of variation is defined as

$$\text{Coeff. Variation} = \frac{\sqrt{\sigma^2}}{\mu}$$

Recall that the second moment about the mean is equal to variance, σ^2 and the first moment about the origin is equal to mean, μ then the coefficient of variation of TE-W-E distribution is given as

Coeff. Variation =

$$\frac{\sqrt{\sum_{p=0}^{s}(-1)^p\binom{2}{p}\mu^p\left((1-\omega)B\left(\tau-\frac{2-p}{\alpha},\frac{2-p}{\alpha}+1\right)+\ 2\omega B\left(2\tau-\frac{2-p}{\alpha},\frac{2-p}{\alpha}+1\right)\right)\tau\delta^{\frac{2-p}{\alpha}}}}{\tau\delta^{1/\alpha}\left((1-\omega)B\left(\tau-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+\ 2\omega B\left(2\tau-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)\right)}$$

3.5 Coefficient of Skewness for TE-W-E Distribution

The coefficient of skewness is defined as

$$\text{Coeff. Skewness} = \frac{E(x-\mu)^3}{(E(x-\mu)^2)^{\frac{3}{2}}}$$

Therefore, the coefficient of skewness for TE-W-E is given as

Coeff. Skewness =

$$\frac{\sum_{p=0}^{s}(-1)^p\binom{3}{p}\mu^p\left((1-\omega)B\left(\tau-\frac{3-p}{\alpha},\frac{3-p}{\alpha}+1\right)+\ 2\omega B\left(2\tau-\frac{3-p}{\alpha},\frac{3-p}{\alpha}+1\right)\right)\tau\delta^{\frac{3-p}{\alpha}}}{\left(\sum_{p=0}^{\infty}(-1)^p\binom{2}{p}\mu^p\left((1-\omega)B\left(\tau-\frac{2-p}{\alpha},\frac{2-p}{\alpha}+1\right)+\ 2\omega B\left(2\tau-\frac{2-p}{\alpha},\frac{2-p}{\alpha}+1\right)\right)\tau\delta^{\frac{2-p}{\alpha}}\right)^{\frac{3}{2}}}$$

3.6 Coefficient of Kurtosis forTE-W-E Distribution

The coefficient of kurtosis is defined as

$$\text{Coeff. Kurtosis} = \frac{E(x-\mu)^4}{(E(x-\mu)^2)^2}$$

Therefore, the coefficient of kurtosis is given as

Coeff. Kurtosis =

$$\frac{\sum_{p=0}^{s}(-1)^p\binom{4}{p}\mu^p\left((1-\omega)B\left(\tau-\frac{4-p}{\alpha},\frac{4-p}{\alpha}+1\right)+\ 2\omega B\left(2\tau-\frac{4-p}{\alpha},\frac{4-p}{\alpha}+1\right)\right)\tau\delta^{\frac{4-p}{\alpha}}}{\left(\sum_{p=0}^{\infty}(-1)^p\binom{2}{p}\mu^p\left((1-\omega)B\left(\tau-\frac{2-p}{\alpha},\frac{2-p}{\alpha}+1\right)+\ 2\omega B\left(2\tau-\frac{2-p}{\alpha},\frac{2-p}{\alpha}+1\right)\right)\tau\delta^{\frac{2-p}{\alpha}}\right)^{2}}$$

3.7 The Moment Generating Function (MGF) of TE-W-E Distribution

The moment generating function (mgf) of TE-W-E distribution is defined as

$$M_x(t) = E(e^{tx})$$

Recall that by power series expansion

$$e^{tx} = \sum_{s=0}^{\infty}\frac{(tx)^s}{s!} = \sum_{k=0}^{\infty}\frac{t^s x^s}{s!}$$

$$\Rightarrow E(e^{tx}) = E\left(\sum_{s=0}^{\infty}\frac{t^s x^s}{s!}\right) = \sum_{s=0}^{\infty}\frac{t^s}{s!}E(x^s)$$

where;

$$E(x^s) =$$

$$\tau\delta^{s/\alpha}\left(\frac{(1-\omega)\Gamma\left(\tau-\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}+1\right)}{\Gamma(\tau+1)} + \frac{2\omega\Gamma\left(2\tau-\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}+1\right)}{\Gamma(2\tau+1)}\right)$$

$$M_x(t) = E(e^{tx})$$

$$= \sum_{s=0}^{\infty}\frac{t^s\delta^{s/\alpha}\tau}{s!}\left(\frac{(1-\omega)\Gamma\left(\tau-\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}+1\right)}{\Gamma(\tau+1)} + \frac{2\omega\Gamma\left(2\tau-\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}+1\right)}{\Gamma(2\tau+1)}\right)$$

$$M_x(t) =$$

$$\sum_{s=0}^{\infty}\frac{t^s\delta^{\frac{s}{\alpha}}\tau}{s!}\left((1-\omega)B\left(\tau-\frac{s}{\alpha},\frac{s}{\alpha}+1\right)+\ 2\omega B\left(2\tau-\frac{s}{\alpha},\frac{s}{\alpha}+1\right)\right)$$

3.8 Survival Analysis

3.8.1 The Survival Function of TE-W-E Distribution

The survival function of TE-W-E distribution is given as

$$S(x) = 1 - F(x;\ \tau,\omega,\alpha,\delta)$$

$$S(x) = 1 - \left\{\left(1-\left(\frac{\delta}{x^\alpha+\delta}\right)^\tau\right)\left(1+\omega\left(\frac{\delta}{x^\alpha+\delta}\right)^\tau\right)\right\}$$

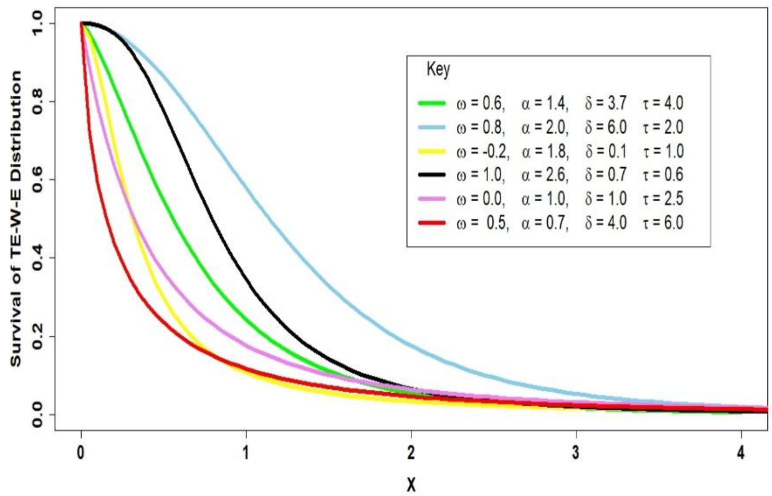

Figure 3: Plot of survival function for some arbitrary parameter values of TE-W-E distribution

From the graph in Figure 3, we can see that the value of the survival function equals 1 at initial time and it decreases as time on *x*-axis increases and equals zero as time on *x*-axis turns to be larger. This means that the TE-W-E distribution be appropriate in modeling time or age-dependent events, where survival or reliability decreases with time.

3.8.2 The Hazard Function of TE-W-E Distribution

The hazard function of TE-W-E distribution is given as

$$h(x) = \frac{f(x;\tau,\omega,\alpha,\delta)}{1-F(x;\tau,\omega,\alpha,\delta)} = \frac{f(x;\tau,\omega,\alpha,\delta)}{S(x)}$$

$$h(x) = \frac{\left(\frac{\alpha\delta^\tau x^{\alpha-1}}{(x^\alpha+\delta)^{\tau+1}}\tau(1-\omega) + \frac{2\omega\tau\alpha\delta^{2\tau}x^{\alpha-1}}{(x^\alpha+\delta)^{\ 2\tau+1}}\right)}{1-\left\{\left(1-\left(\frac{\delta}{x^\alpha+\delta}\right)^\tau\right)\left(1+\omega\left(\frac{\delta}{x^\alpha+\delta}\right)^\tau\right)\right\}}$$

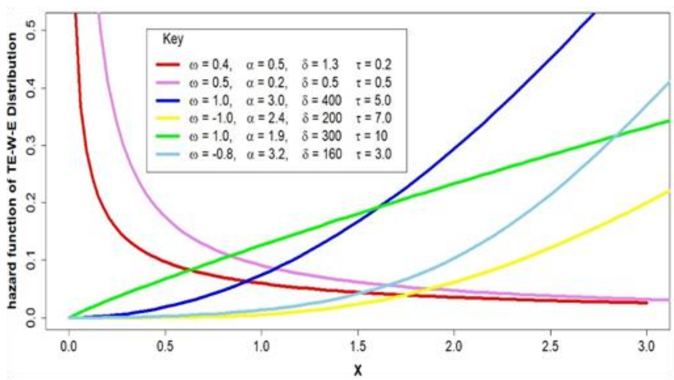

Figure 4: Plot of hazard function for some arbitrary parameter values of TE-W-E distribution

From the graph in Figure 4, we can see that the hazard function of the proposed TE-W-E distribution has J-shaped or bathtub-shaped. The J-shaped or bathtub-shaped of the hazard function of the proposed TE-W-E distribution shows that it is highly flexible to model any positively skewed data. It can also be seen that the value of the hazard function increases when time on *x*-axisincreases. It gets higher as the value of time increases and decreases thereafter to constant. The exhibition of different shapes of the hazard function of the TE-W-E distribution makes it suitable in modeling time or age-dependent events, where hazard decreases, increases or remains constant with time.

3.8.3 Reversed Hazard Function of TE-W-E distribution

The reversed hazard function of TE-W-E distribution is given as

$$r(x) = \frac{f(x;\tau,\omega,\alpha,\delta)}{F(x;\tau,\omega,\alpha,\delta)}$$

$$r(x) = \frac{\left(\frac{\tau\alpha\delta^{\tau}x^{\alpha-1}}{(x^{\alpha}+\delta)^{\tau+1}}(1-\omega) + \frac{2\omega\tau\alpha\delta^{2\tau}x^{\alpha-1}}{(x^{\alpha}+\delta)^{2\tau+1}}\right)}{\left\{\left(1-\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\left(1+\omega\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\right\}}$$

3.8.4 Odd Function of TE-W-E Distribution

The odd function of TE-W-E distribution is given as

$$ODD(x) = \frac{F(x;\tau,\omega,\alpha,\delta)}{S(x)}$$

$$ODD(x) = \frac{\left\{\left(1-\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\left(1+\omega\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\right\}}{1-\left\{\left(1-\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\left(1+\omega\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\right\}}$$

3.8.5 Cumulative Hazard Function of TE-W-E Distribution

The cumulative hazard function of TE-W-E distribution is given as

$$H(x) = -ln[S(x)]$$

$$H(x) = ln\left[1-\left\{\left(1-\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\left(1+\omega\left(\frac{\delta}{x^{\alpha}+\delta}\right)^{\tau}\right)\right\}\right]$$

3.8.6 The Quantile Function of TE-W-E Distribution

Assume X is a random variable that follows TE-W-E distribution with parametersτ,ω,α and δ; that is,

$X \sim$ TE-W-E $(\tau,\omega,\alpha,\delta)$; then the quantile function of TE-W-E distribution is given as

$$x_q = \delta^{\frac{1}{\alpha}}\left\{\left(\frac{(\omega-1)\pm\sqrt{(\omega-1)^2+4\omega(1-q)}}{2\omega}\right)^{-\frac{1}{\tau}} - 1\right\}^{\frac{1}{\alpha}}, \quad \text{for } \omega \neq 0$$

A random variable X from TE-W-E$(\tau,\omega,\alpha,\delta)$ can be generated when we assume that q has uniform distribution over the interval (0, 1); i.e., $q \sim U(0,1)$

3.8.7 The Median of TE-W-E Distribution

The median of TE-W-E distribution is obtained when we substitute for $q = 0.5$ in the quantile function of TE-W-E distribution. Therefore, the median of TE-W-E distribution is given by

$$x_{0.5} = \delta^{1/\alpha}\left\{\left(\frac{(\omega-1)\pm\sqrt{(\omega-1)^2+2\omega}}{2\omega}\right)^{-\frac{1}{\tau}} - 1\right\}^{1/\alpha}, \quad for\ \omega \neq 0$$

3.8.8 The Rényi Entropy of the TE-W-E Distribution

Entropy is a mathematical measure of the degree of randomness or disorderedness in a set of data. In other words, entropy is a measure of variation or uncertainty of a random variable. The idea of entropy has been used in different areas such as statistics, queuing theory and reliability estimation. This entropy was introduced by Claude Shannon (Shannon, 1948; Cover & Thomas, 1991).

Let X be a random variable with the TE-W-E distribution with parameters $\tau,\omega,\alpha,$ and δ; then the Rényi Entropy of a random variable X that follows TE-W-E distribution is given by

$$I_R(X) = \frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty}(f(x))^{\rho}\,dx\right\}, \qquad \rho > 0\ and\ \rho \neq 0$$

$$I_R(X) = \frac{1}{1-\rho}log\left\{\alpha^{\rho-1}\delta^{\left\{\frac{(\alpha-1)\rho-\alpha+1}{\alpha}\right\}+1}\left(\frac{\tau}{\delta}\right)^{\rho}\left(\frac{\Gamma\left((\tau+1)-\left(\frac{(\alpha-1)\rho-\alpha+1}{\alpha}\right)\right)}{\Gamma(\tau+1)}\right)^{\rho} * \sum_{s,m=0}^{\infty} w_{s,m,\rho}\,(2\rho-s)^{-\left(\left\{\frac{(\alpha-1)\rho-\alpha+1}{\alpha}\right\}+1\right)}\Gamma\left(\left\{\frac{(\alpha-1)\rho-\alpha+1}{\alpha}\right\}+1\right)\right\}$$

where

$$w_{s,m,\rho} = (-1)^m \frac{\Gamma(\rho+1)\Gamma(s+1)}{s!m!\Gamma(\rho+1-s)\Gamma(s+1-m)} \left(\frac{\Gamma\left((\tau+1) - \left(\frac{(\alpha-1)\rho-\alpha+1}{\alpha}\right)\right)}{\Gamma(\tau+1)} \right)^{\rho-s} \omega^{m+\rho-s} 2^{\rho-s}$$

.

3.8.9 The Order Statistic of TE-W-E Distribution

Let $x_1, x_2, \cdots, x_n$ denote the order statistics of random samples $y_1, y_2, \cdots, y_n$ from TE-W-E distribution population with cumulative density function (cdf) $F_y(y;\ \tau, \omega, \alpha, \delta)$ and probability density function (pdf) $f_y(y;\ \tau, \omega, \alpha, \delta)$; the density function of the sth order statistic where $1 \le s \le n$ is defined as

$$f_{s:n}(x) = \frac{f(x)}{B(k, n-s+1)} [F(x)]^{s-1} [1 - F(x)]^{n-s}$$

$$f_{s:n}(x) = \frac{n!\, \alpha x^{\alpha-1}}{(s-1)!\,(n-s)!\,(x^\alpha + \delta)} \tau \left((1-\omega) + 2\omega \left(\frac{\delta}{(x^\alpha + \delta)} \right)^\tau \right) Z_{d,f,g,h,j}$$

where,

$$Z_{d,f,g,h,j} = \sum_{d,f,g,h,j=0}^{\infty} (-1)^{d+g+h} \frac{\Gamma(s)\Gamma(s)\Gamma(n-s+1)\Gamma(g+1)\Gamma(g+1) \left(\frac{\delta}{(x^\alpha+\delta)}\right)^{\tau(1+d+f+h+j)} \omega^{f+j}}{d!\,f!\,g!\,h!\,j!\,\Gamma(s-d)\Gamma(s-f)\Gamma(n-s-g+1)\Gamma(g-h+1)\Gamma(g-j+1)}$$

4. PARAMETERS ESTIMATION OF THE TE-W-E DISTRIBUTION USING MAXIMUM LIKELIHOOD ESTIMATION (MLE) METHOD

Let $x_1, x_2, \cdots, x_n$ be the random sample of size n from the TE-W-E distribution with unknown parameters $\omega, \alpha, \tau,$ and δ. Let $\Theta = (\omega, \alpha, \tau, \delta)^T$ be the 4 x 1 parameter vector.

For determining the MLE of Θ, we consider likelihood function as follows

$$L(\Theta) = \prod_{i=1}^{n} f(x_i;\ \tau, \omega, \alpha, \delta)$$

$$= \alpha^n \delta^{-n} \tau^n \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (x_i^\alpha \delta^{-1} + 1)^{-(\tau+1)} \prod_{i=1}^{n} (1 - \omega + 2\omega(x_i^\alpha \delta^{-1} + 1)^{-\tau})$$

Taking the log-likelihood, we have

$$\log L(\Theta) = n\log\alpha - n\log\eta + n\log\tau + (\alpha-1)\sum_{i=1}^{n} \log x_i - (\tau+1)\sum_{i=1}^{n} \log(x_i^\alpha \delta^{-1} + 1) + \sum_{i=1}^{n} \log(1 - \omega + 2\omega(x_i^\alpha \delta^{-1} + 1)^{-\tau}) \quad (35)$$

Differentiating Eq. (35) with respect to the parameters $\omega, \alpha, \tau,$ and δ respectively, and equating each of the resulting Eq. to zero, we have

$$D_\omega = \frac{d\log L(\Theta)}{d\omega} = \sum_{i=1}^{n} \frac{2(x_i^\alpha \delta^{-1} + 1)^{-\tau} - 1}{\left(1 - \omega + 2\omega\left(x_i^\alpha \delta^{-1} + 1\right)^{-\tau}\right)} = 0$$

$$D_\alpha = \frac{d\log L(\Theta)}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log x_i - (\tau+1)\delta^{-1} \sum_{i=1}^{n} \frac{x_i^\alpha \ln x_i}{(x_i^\alpha \delta^{-1} + 1)} - 2\omega\tau\delta^{-1} \sum_{i=1}^{n} \frac{x_i^\alpha \ln x_i\, (x_i^\alpha \delta^{-1} + 1)^{-(\tau+1)}}{(1 - \omega + 2\omega(x_i^\alpha \delta^{-1} + 1)^{-\tau})} = 0$$

$$D_\tau = \frac{d\log L(\Theta)}{d\tau} = \frac{n}{\tau} - \sum_{i=1}^{n} \log(x_i^\alpha \delta^{-1} + 1) + 2\omega \sum_{i=1}^{n} \frac{(x_i^\alpha \delta^{-1} + 1)^{-\tau} \ln(x_i^\alpha \delta^{-1} + 1)}{\left(1 - \omega + 2\omega\left(x_i^\alpha \delta^{-1} + 1\right)^{-\tau}\right)} = 0$$

$$D_\delta = \frac{d\log L(\Theta)}{d\delta} = (\tau+1)\delta^{-2} \sum_{i=1}^{n} \frac{x_i^\alpha (x_i^\alpha \delta^{-1} + 1)^{-1}}{\left(x_i^\alpha \delta^{-1} + 1\right)} + 2\omega\tau\delta^{-2} \sum_{i=1}^{n} \frac{x_i^\alpha (x_i^\alpha \delta^{-1} + 1)^{-(\tau+1)}}{\left(1 - \omega + 2\omega\left(x_i^\alpha \delta^{-1} + 1\right)^{-\tau}\right)} - \frac{n}{\delta} = 0$$

By setting the nonlinear system of Eq.s to zero, that is, $D_\omega = D_\alpha = D_\tau = D_\delta = 0$ and solving them simultaneously gives the MLE $\hat{\Theta} = \left(\hat{\omega},\ \hat{\alpha}, \hat{\tau}, \hat{\delta},\right)^T$. Moreover, numerical computations are needed to find the solution of this nonlinear system of Eq.s, $D_\omega = D_\alpha = D_\tau = D_\delta = 0$ and it is usualy more convenient to obtain by using nonlinear optimization methods such as the Newton-Rapson algorithm in R statistical software through optim function in the software (R Core Team, 2024).

5. SIMULATION STUDY

5.1 Simulation Study to Evaluate the Behavior of Median, Mean, Variance, Skewness and Kurtosis of TE-W-E Model

A simulation study was used to examine the effect of each parameter of the TE-W-E model on the median, mean, variance, skewness and kurtosis of the model. In this simulation study, sample size of 1000 was generated and it was done using some arbitrary values of each parameters of the TE-W-E model. In addition, this simulation study would be used to select the best arbitrary parameter values for assessing the performance of the maximum likelihood (ML) estimator on the estimation of the proposed TE-W-E distribution parameters.

5.1.2 Behaviour of the TE-W-E Model under Different Values of Transmuted Parameter (ω)

Table 1 presents the effect of the transmuted parameter $\boldsymbol{\omega}$, on the proposed TE-W-E distribution for different values of the $\boldsymbol{\omega}$, while holding the scale parameter $\boldsymbol{\tau = 3}$, the shape parameter $\boldsymbol{\alpha = 1.1}$, and the compounding parameter $\boldsymbol{\delta = 9}$ constant. The descriptive statistics reported include the minimum, first quartile, median, third quartile, maximum, mean, variance, standard deviation, CV, coefficient of skewness, and coefficient of kurtosis. The purpose of this analysis is to examine the impact of the $\boldsymbol{\omega}$ on the distributional characteristics of the TE-W-E model.

The results in Table 1 show that both the mean and median decrease monotonically as ω increases from −1.0 to 1.0. When $\omega = -1.0$, the mean is 5.91 with a corresponding median of 4.05, whereas at $\omega = 1.0$, the mean reduces substantially to 1.68 and the median to 1.12. This behavior demonstrates that the ω strongly effects the location of the distribution, shifting the mass of the distribution from higher values toward lower values as ωincreases. Therefore, negative values of ω are associated with heavier right tails and larger central values, while positive values contract the distribution toward smaller outcomes. The variability of the TE-W-E distribution is also sensitive to changes in ω.

Table 1: Median, mean, variance, CV, skewness and kurtosis for selected values of transmuted parameter $\boldsymbol{\omega}$, of theTE-W-E model. Parameters: $\boldsymbol{\tau = 3,\ \alpha = 1.1,\ \delta = 9.}$

ω	Min.	First Quantile	Median	Third Quantile	Max.	Mean	Variance	Std. Dev.	CV	Coef. Skewness	Coef. Kurtosis
-1.0	0.2601	2.2997	4.0482	7.0258	73.0546	5.9129	41.5670	6.4472	1.0904	4.0646	26.8023
-0.9	0.1442	2.1400	3.8668	6.8040	71.8374	5.7036	40.3758	6.3542	1.1141	4.0595	26.7448
-0.8	0.0919	1.9828	3.6826	6.5756	70.5746	5.4938	39.1056	6.2534	1.1383	4.0603	26.7435
-0.7	0.0664	1.8298	3.4962	6.3406	69.2617	5.2835	37.7534	6.1444	1.1629	4.0675	26.8037
-0.6	0.0519	1.6827	3.3084	6.0987	67.8937	5.0729	36.3189	6.0265	1.1880	4.0811	26.9285
-0.5	0.0426	1.5434	3.1202	5.8499	66.4646	4.8619	34.8021	5.8993	1.2134	4.1014	27.1213
-0.4	0.0362	1.4133	2.9330	5.5942	64.9672	4.6507	33.2030	5.7622	1.2390	4.1283	27.3866
-0.3	0.0315	1.2937	2.7482	5.3320	63.3931	4.4391	31.5218	5.6144	1.2648	4.1621	27.7298
-0.2	0.0280	1.1850	2.5676	5.0639	61.7321	4.2273	29.7583	5.4551	1.2904	4.2032	28.1573
-0.1	0.0251	1.0872	2.3929	4.7908	59.9713	4.0152	27.9128	5.2833	1.3158	4.2517	28.6768
0.1	0.0209	0.9227	2.0681	4.2360	56.0825	3.5906	23.9772	4.8967	1.3638	4.3716	30.0241
0.2	0.0194	0.8542	1.9207	3.9590	53.9075	3.3781	21.8886	4.6785	1.3850	4.4428	30.8664
0.3	0.0180	0.7936	1.7844	3.6863	51.5339	3.1655	19.7211	4.4408	1.4029	4.5204	31.8235
0.4	0.0168	0.7399	1.6594	3.4215	48.9112	2.9528	17.4770	4.1805	1.4158	4.6016	32.8799
0.5	0.0158	0.6923	1.5457	3.1680	45.9647	2.7400	15.1593	3.8935	1.4210	4.6802	33.9806
0.6	0.0149	0.6498	1.4425	2.9292	42.5775	2.5270	12.7732	3.5740	1.4143	4.7417	34.9696
0.7	0.0141	0.6120	1.3494	2.7073	38.5481	2.3137	10.3281	3.2137	1.3890	4.7508	35.4160
0.8	0.0134	0.5780	1.2654	2.5038	33.4802	2.1001	7.8440	2.8007	1.3336	4.6138	34.0781
0.9	0.0127	0.5474	1.1896	2.3191	26.4228	1.8863	5.3742	2.3182	1.2290	4.0607	27.2924
1.0	0.0122	0.5198	1.1213	2.1526	15.5263	1.6765	3.1800	1.7832	1.0636	2.6621	10.9102

The variance decreases steadily from 41.57 at $\omega = -1.0$ to 3.18 at $\omega = 1.0$, a trend that is consistent with the observed reduction in standard deviation over the same range of values from 6.45 to 1.78. This shows that larger positive values of ωresult in a more concentrated distribution around the mean, while negative values of ω correspond to greater variance. The CV exceeds the value 1 across all values of ω.Conversely, the CV also decreases with increasing ω, ranging from 1.32 at $\omega = -0.1$ to 1.06 at $\omega = 1.0$. This suggests that as the transmuted parameter increases, the distribution becomes less dispersed compare to its mean. The coefficient of skewness remains positive throughout the ranges of ω and it is showing that the TE-W-E distribution is consistently right-skewed irrespective of the value of ω. Nevertheless, the degree of skewness is not constant. It increases as ω increases from negative values, reaching a peak of approximately 4.75 at $\omega = 0.7$, before decreasing to 2.66 when$\omega = 1.0$. This shows that moderate positive values of ω increase asymmetry, whereas extreme positive values of it counteract this effect; thereby producing a distribution that is still right-skewed but less than in the mid-range of ω. The TE-W-E distribution exhibits obviously high kurtosis for most values of ω, with coefficients of kurtosiswell above 3, the benchmark for the normal distribution. Excess kurtosis values ranging between 26 and 34 are observed when $-1.0 \leq \omega \leq 0.7$ and it shows that the TE-W-E distribution is highly leptokurtic, with extreme peakedness and heavy tails. However, as ω approaches 1.0, the coefficient of kurtosis decreases to 10.91and it shows that the tails of the TE-W-E distribution become thinner and the influence of extreme observations diminishes. This transition shows the flexibility of the TE-W-E model in capturing both heavily tailed and moderately tailed behavior, depending on the value of ω. The minimum remains small and stable across all values of ω, ranging between 0.01 and 0.26, which reflects the distribution's bounded lower support. In contrast, the maximum decreases sharply from 73.05 at $\omega = -1.0$ to 15.53 when$\omega = 1.0$. This significant contraction of the maximum value as ω increases again illustrates the role of the transmuted parameter in moderating the heaviness of the right tail. The findings shows that negative values of ω make the proposed TE-W-E distribution to produce larger mean and median, greater variance, stronger heavy-tailedness, and extreme right skewness while positive values of ωgradually reduce the median and mean, variability, and maximum values, thereby producing a more concentrated distribution with lighter tails. Across all values, the proposed TE-W-E distribution remains positively skewed and leptokurtic, although the intensity of these characteristics diminishes as $\omega \rightarrow 1.0$. Hence, the ω serves as a powerful shape regulator that can tailor the distribution between heavy-tailed, highly skewed forms and more compact, moderately skewed alternatives. This flexibility enhances the suitability of the TE-W-E model in modeling different real-world datasets, especially those exhibiting varying degrees of skewness and kurtosis.

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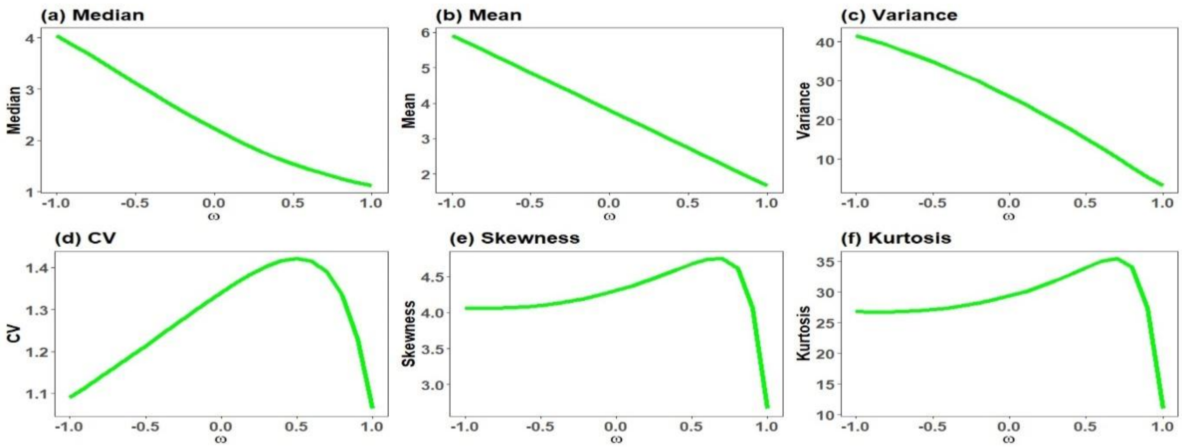

Figure 5: Plots of the median, mean, variance, CV, skewness, and kurtosis of the TE-W-E distribution against ω for fixed $\tau = 3$, $\alpha = 1.1$and $\delta = 9$

The plots in Figure 5 complements the empirical results in Table 1 which display the variation of median, mean, variance, CV, skewness, and kurtosis of the TE-W-E distribution with respect to ω. Figure 5a and Figure 5b show that both median and mean decrease monotonically as ω increases. It can be noticed that the mean values consistently exceeding the median values and this implies that TE-W-E distribution is persistent right-skewness. Figure 5c shows that the variance is large for negative ω values, indicating heavy-tailed behaviour, but gradually decreases as ω increases, indicating reduced spread. Figure 5d shows the CV initially increases with ω, reaching a peak around moderate positive values, before decreasing, suggesting that relative variability is maximized in this region. Figure 5e shows that the skewness remains positive across all values of ω, confirming right-skewness of the TE-W-E distribution, and increases with positive ω, showing stronger asymmetry as the TE-W-E distribution contracts. Figure 5f shows that the kurtosis is obviously high at negative ω, showing leptokurtic and heavy-tailed behaviour, but decreases with ω, converging toward more moderate peakedness for positive values of ω.

5.1.3 Behaviour of the TE-W-E Model under Different Values of Shape Parameter (α)

Table 2 presents the effect of the shape parameter $\boldsymbol{\alpha}$, on the proposed TE-W-E distribution for different selected values of $\boldsymbol{\alpha}$, under the fixed values of the transmuted parameter $\boldsymbol{\omega} = \mathbf{0.8}$, **the** scale parameter $\boldsymbol{\tau} = \mathbf{2}$, and the compounding parameter $\boldsymbol{\delta} = \mathbf{9}$. The descriptive statistics reported include the minimum, first quartile, median, third quartile, maximum, mean, variance, standard deviation, CV, coefficient of skewness, and coefficient of kurtosis. The purpose of this analysis is to investigate the effect of the $\boldsymbol{\alpha}$ on the proposed TE-W-E distributional features.

Table 2: Median, mean, variance, CV, skewness and kurtosis for some selected values of transmuted parameter$\boldsymbol{\alpha}$, of the TE-W-E model.
Parameters: $\boldsymbol{\tau} = \mathbf{2},\ \boldsymbol{\omega} = \mathbf{0.8},\ \boldsymbol{\delta} = \mathbf{7}$

$\boldsymbol{\alpha}$	Min.	First Quantile	Median	Third Quantile	Max.	Mean	Variance	Std. Dev.	CV	Coef. Skewness	Coef. Kurtosis
0.6	0.0000	0.4606	2.0910	7.4331	1183.359	14.1100	4284.618	65.4570	4.6391	11.6994	167.1597
1.0	0.0009	0.6281	1.5567	3.3320	69.8023	3.0012	28.8911	5.3750	1.7910	6.2468	54.5941
1.2	0.0028	0.6787	1.4460	2.7264	34.4000	2.2504	9.0550	3.0091	1.3371	4.7590	33.7314
1.6	0.0122	0.7477	1.3187	2.1217	14.2043	1.6623	2.2245	1.4915	0.8973	3.0459	15.2817
2.1	0.0348	0.8013	1.2346	1.7738	7.5515	1.3922	0.8154	0.9030	0.6486	1.9903	7.2611
2.6	0.0664	0.8362	1.1856	1.5887	5.1189	1.2691	0.4261	0.6527	0.5143	1.4132	4.1169
2.8	0.0806	0.8470	1.1712	1.5370	4.5554	1.2375	0.3471	0.5891	0.4760	1.2492	3.3981
3.0	0.0953	0.8564	1.1590	1.4936	4.1174	1.2118	0.2889	0.5375	0.4436	1.1099	2.8512
3.2	0.1104	0.8647	1.1483	1.4566	3.7689	1.1907	0.2447	0.4947	0.4155	0.9899	2.4278
3.6	0.1410	0.8788	1.1308	1.3970	3.2523	1.1581	0.1829	0.4277	0.3693	0.7930	1.8315
3.9	0.1639	0.8876	1.1202	1.3615	2.9702	1.1396	0.1512	0.3889	0.3413	0.6733	1.5298
4.3	0.1939	0.8975	1.1084	1.3230	2.6842	1.1203	0.1207	0.3475	0.3102	0.5404	1.2503
4.6	0.2158	0.9038	1.1010	1.2991	2.5168	1.1088	0.1037	0.3221	0.2905	0.4561	1.1035
5.4	0.2709	0.9175	1.0854	1.2497	2.1951	1.0861	0.0730	0.2702	0.2488	0.2767	0.8717
8.0	0.4141	0.9435	1.0569	1.1624	1.7001	1.0499	0.0321	0.1790	0.1705	-0.0654	0.7418
9.0	0.4567	0.9496	1.0504	1.1431	1.6028	1.0427	0.0252	0.1588	0.1523	-0.1468	0.7727

Table 2 shows that both the median and mean decrease gradually as α increases. For small values of $\alpha = 0{,}6$, it can be seen that the mean value of 14.1100 is far exceeded the median value of 2.0910. This indicates that there is a strong right-skewness. The results show that as α increases, the mean and median values are converging to 1, indicating that theTE-W-E becomes symmetric as α increases. The results also show that the mean values are smaller than the median values at $\alpha = 8$ and $\alpha = 0.9$ and this implies that the TE-W-E distribution skews to the left at very larger values of α.

The variance and CV show a dramatic decrease as α increases. When $\alpha = 0.6$, the variance is extremely large with value of 4284.62 and CV value is 4.6391.This shows heavy dispersion and instability. However, when $\alpha = 9$, the variance decreases to 0.0252 and CV decreases to 0.1523 and it shows that higher values of α produce a well concentrated distribution with minimal variability.

The skewness begins at very high positive values. When $\alpha = 0.6$, the coefficient of skewness is 11.6994 and decreases consistently as α increases. This shows a transition from highly skewed distributions to nearly symmetric ones. When α = 8 and $\alpha = 9$, the coefficient of skewness becomes slightly negative with the values of -0.0654 and -0.1468, respectively. This suggests that the TE-W-E distribution is left-skewed in the some case as α becomes large.

Similarly, the kurtosis starts at an extremely high value. When $\alpha = 0.6$, coefficient of kurtosis is 167.1597, showing that TE-W-E distribution is heavy-tailed and leptokurtic at this value of α. As α increases, kurtosis decreases monotonically, moving toward the values close to 1 for large α, showing platykurtic behavior and thinner tails.

The results in Table 2 shows that the parameter α plays an important role in shaping the proposedTE-W-E distribution. Low values of α produce heavy-tailed, highly dispersed, and strongly right-skewed distributions while higher values of α produces light-tailed, symmetric or left-skewed distributions with low variability. This dynamic behavior demonstrates the flexibility of the proposed TE-W-E distribution in accepting diverse empirical data patterns such as highly skewed, heavy-tailed, and symmetric datasets.

The plots in Figure 6 complements the empirical results in Table 2 which shows the effect of the shape parameter α of the proposed TE-W-E distribution on the median, mean, variance, CV, skewness, and kurtosis. Figure 6a and Figure 6b show that both median and mean decrease sharply as α increases and thereafter, stabilize and converge towards 1.0 before approximately to zero. The discrepancy between mean and median values reduces, making the distribution moving towards symmetry. Figure 6c shows that the variance displays an obvious downward trend, starting from extremely high levels when α is small which shows heavy-tailed behavior and tapering off as α increases. Figure 5d shows that the CV moves from very high levels at small values of α to approximately constant. The CV decreases to very low levels at large values of α. Figure 5e shows that the skewness decreases gradually as α increases.The coefficient of skewness is strongly positive when α is small, indicating a heavy right tail and asymmetry in the distribution. As αincreases, the skewness valuesapproach zero.Figure 5f shows that the coefficient of kurtosis starts at very high values when α is small and this corresponds to leptokurtic, that is, heavy-tailed and peaked behavior. As α increases, the kurtosis decreases sharplyand approaches values near one and this suggests platykurtic or mesokurtic behavior. This movement implies that the tails of the distribution become lighter asα increases, supporting the observed trend towards normal-like behaviour.

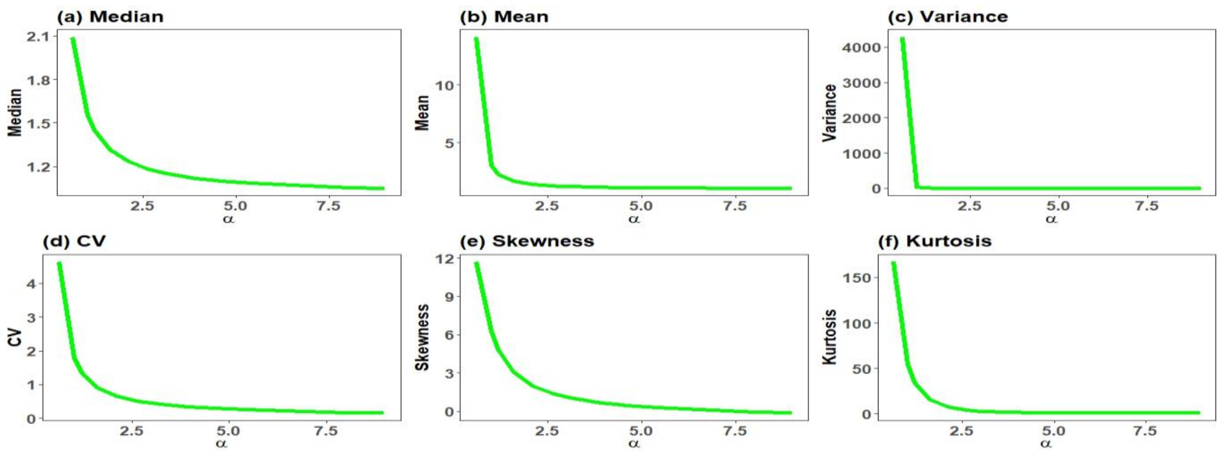

Figure 6: Plots of the median, mean, variance, CV, skewness, and kurtosis of the TE-W-E distribution againstα for fixed $\omega = 0.8$, $\tau = 2$ and $\delta = 7$

5.1.4 Behaviour of the TE-W-E Model under Different Values of Scale Parameter (τ)

Table 3 presents the effect of the scale parameter $\boldsymbol{\tau}$, on the proposed TE-W-E distribution for different selected values of $\boldsymbol{\tau}$, under the fixed values of the transmuted parameter $\boldsymbol{\omega} = \mathbf{0.1}$, **the** shape parameter $\boldsymbol{\alpha} = \mathbf{2.2}$, and the compounding parameter $\boldsymbol{\delta} = \mathbf{10}$. The descriptive statistics reported include the minimum, first quartile, median, third quartile, maximum, mean, variance, standard deviation, CV, coefficient of skewness, and coefficient of kurtosis. The purpose of this analysis is to investigate the effect of the $\boldsymbol{\tau}$ on the proposed TE-W-E distributional characteristics.

Table 3: Median, mean, variance, CV, skewness and kurtosis for some selected values of scale parameter$\boldsymbol{\tau}$, of the TE-W-E model. Parameters: $\boldsymbol{\omega} = \mathbf{0.1}, \boldsymbol{\alpha} = \mathbf{2.2},\ \boldsymbol{\delta} = \mathbf{10}.$

τ	Min.	First Quantile	Median	Third Quantile	Max.	Mean	Variance	Std.Dev.	CV	Coef. Skewness	Coef. Kurtosis
1.6	0.1434	1.365	2.1428	3.2926	18.6221	2.6048	3.7365	1.933	0.7421	2.511	10.8392
1.9	0.1326	1.254	1.9496	2.9401	13.7346	2.2994	2.4445	1.5635	0.6800	2.0417	7.2034
2.2	0.1241	1.1675	1.8025	2.6823	10.9646	2.0835	1.7824	1.3351	0.6408	1.7424	5.217
2.6	0.115	1.0771	1.6517	2.4269	8.7559	1.8747	1.2941	1.1376	0.6068	1.4825	3.71
3.4	0.1018	0.9477	1.4409	2.0833	6.4496	1.6002	0.8269	0.9093	0.5682	1.1882	2.2575
3.8	0.0968	0.8991	1.363	1.9602	5.7728	1.5032	0.6993	0.8362	0.5563	1.0974	1.8653
4	0.0946	0.8776	1.3288	1.9067	5.4993	1.4612	0.6492	0.8057	0.5514	1.0602	1.7125
5	0.0854	0.7903	1.1911	1.6945	4.5254	1.2959	0.4782	0.6915	0.5336	0.9262	1.1992

From Table 3, it can be seen that both the median and mean decrease consistently as τ increases from 1.6 to 5.0. As the values of τ decreases, the mean values keep being more than the median values and this implies that the TE-W-E distribution is positively skewed as the values of τ increase.

As τ increases, the mean and median converge, indicating a decrease in skewness and a shift toward greater symmetry. The variance reduces sharply as τ increases. When $\tau = 1.6$, the variance is 3.7365, but when$\tau = 5.0$, it falls to 0.4782. This decrease indicates that there is an increasing concentration of the distribution around its median and mean as scale values get large. Similarly, the CV falls from 0.7421 at $\tau = 1.6$ to 0.5336 at $\tau = 5.0$ and this indicates the reduced relative dispersion and tighter clustering of observations. The coefficient of skewness shows a monotonic decrease asτ increases. At initial value of$\tau = 1.6$, the skewness value is 2.511, indicating a strong right-tail dominance. As τ increases, the skewness value steadily decreases to 0.9262 at$\tau = 5.0$. This decrease of τ values show a gradual reduction of asymmetry while the distribution becomes more symmetric with higher values of τ.The coefficient of kurtosis is obviously high at small τ. The kurtosis value is10.8392 at $\tau = 1.6$, showing heavy-tailed and peaked behavior. Asτ values increase, the kurtosis decreases substantially, getting to 1.1992 at $\tau = 5.0$. This shows a transition from leptokurtic to platykurtic behavior. The tails of the distribution become lighter and the peak flattens and this further shows the stabilization effect of τ.

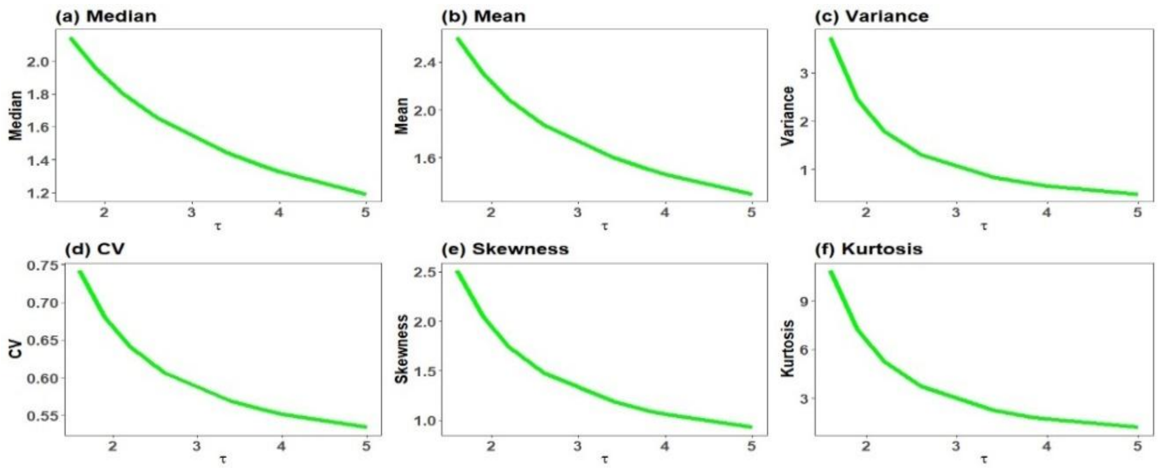

Figure 7: Plots of the median, mean, variance, CV, skewness, and kurtosis of the TE-W-E distribution againstτ for fixed$\omega = 0.1, \alpha = 2.2$and $\delta = 10$

The plots in Figure 7complements the empirical results in Table 3 which shows the effect of the scale parameter τ of the proposed TE-W-E distribution on the median, mean, variance, CV, skewness, and kurtosis. Figure 7a and Figure 7b show that both median and mean decrease monotonically as τ increases. It implies that higher values of τ shift the distribution toward smaller expected outcome. Figure 7c and Figure 7d show that both the variance and CV decrease sharply as τincreases. This means that there is an obvious reduction in dispersion and the graphs display that large λ value provides more concentrated values of TE-W-E distribution. Figure 5e shows that the skewness decreases gradually as τ increases.The coefficient of skewness is moving from strongly positivewhen values ofτ are small toward values closer to symmetry. This implies that the proposed TE-W-E distribution moves from highly positively skewed to moderately skewed as τ increases.Figure 5f shows that the coefficient of kurtosis starts at very high values when τ is small and approximately normal at larger τ values.This reduction suggests a shift from a leptokurtic distribution with heavy tails to one that is flatter and more regular.

5.1.5 Behaviour of the TE-W-E Model under Different Values of Compounding Parameter (δ)

Table 4 presents the effect of the compounding parameter δ, on the proposed TE-W-E distribution for some different selected values of δ, under the fixed values of the transmuted parameter $\boldsymbol{\omega = 0.4}$, the shape parameter $\boldsymbol{\alpha = 1.5}$, and the scale parameter $\boldsymbol{\tau = 4}$. The descriptive statistics reported include the minimum, first quartile, median, third quartile, maximum, mean, variance, standard deviation, CV, coefficient of skewness, and coefficient of kurtosis. The purpose of this analysis is to investigate the effect of the δ on the proposed TE-W-E distributional characteristics.

Table 4: Median, mean, variance, CV, skewness and kurtosis for some selected values of scale parameterδ, of the TE-W-E model. Parameters:$\boldsymbol{\omega = 0.4,\ \alpha = 1.6,\ \tau = 4.}$

$\boldsymbol{\delta}$	Min.	First Quantile	Median	Third Quantile	Max.	Mean	Variance	Std. Dev.	CV	Coef. Skewness	Coef. Kurtosis
3	0.004	0.2995	0.5436	0.9453	5.7825	0.7142	0.3798	0.6163	0.8629	2.1733	7.9247
6	0.0063	0.4754	0.8629	1.5005	9.1791	1.1337	0.9571	0.9783	0.8629	2.1733	7.9247
9	0.0082	0.623	1.1307	1.9662	12.028	1.4856	1.6435	1.282	0.8630	2.1733	7.9247
12	0.01	0.7547	1.3697	2.3819	14.5709	1.7997	2.4118	1.553	0.8629	2.1733	7.9247
17	0.0126	0.952	1.7277	3.0045	18.3795	2.2701	3.8374	1.9589	0.8629	2.1733	7.9247
20	0.014	1.0609	1.9254	3.3483	20.4827	2.5299	4.7659	2.1831	0.8629	2.1733	7.9247
25	0.0163	1.2311	2.2343	3.8853	23.768	2.9356	6.4174	2.5333	0.8630	2.1733	7.9247
30	0.0184	1.3902	2.5231	4.3875	26.8399	3.315	8.1834	2.8607	0.8630	2.1733	7.9247
40	0.0223	1.6841	3.0565	5.315	32.5142	4.0159	12.0093	3.4654	0.8629	2.1733	7.9247
50	0.0259	1.9542	3.5467	6.1676	37.7294	4.66	16.1707	4.0213	0.8629	2.1733	7.9247
70	0.0324	2.4456	4.4386	7.7185	47.217	5.8319	25.326	5.0325	0.8629	2.1733	7.9247
90	0.0383	2.8917	5.2482	9.1263	55.8292	6.8956	35.4073	5.9504	0.8629	2.1733	7.9247

The results from Table 4 show that both the median and mean display a steady increase as δ increases. It can be seen from the results that the median value increases from 0.544 when $\delta = 3$ to 5.248 when $\delta = 90$, while the mean value increases from 0.714 to 6.896 at the same range of δ. This trend indicates the direct scaling influence of δ which systematically shifts the median and mean of the distribution towards higher values.The variance increases significantly from 0.380 when $\delta = 3$ to 35.407 when$\delta = 90$, and the standard deviation increases in parallel, from 0.616 to 5.950. This expansion in spread shows that the δ has a magnifying effect on variability, producing wider distributions as δ becomes larger.The results show that the CV keeps unchanged across all values of δ, maintaining a constant value of approximately 0.863. This constant values of CV suggests that, although the absolute dispersion (variance-to-mean ratio) increases, the relative variability of the distribution with respect to its mean is remained constant. This implies that the distribution stretches proportionally as δ increases, thereby retaining the same degree of variability compare to its mean.The results show that the coefficient of skewness and kurtosis remain constant as δ increases. It can be seen that the coefficient of skewness remains constant at 2.173 across all values of δ, indicating a persistent right-skewed structure of the proposed TE-W-E distribution. In the same way, the coefficient of kurtosis remains at 7.925, indicating that the distribution consistently shows a leptokurtic form, indicating heavy tails and a sharper peak compare to the normal distribution. The constancy of these shape measures implies that δ does not change the shape characteristics of the TE-W-E distribution, but instead serves only as a scale modifier.

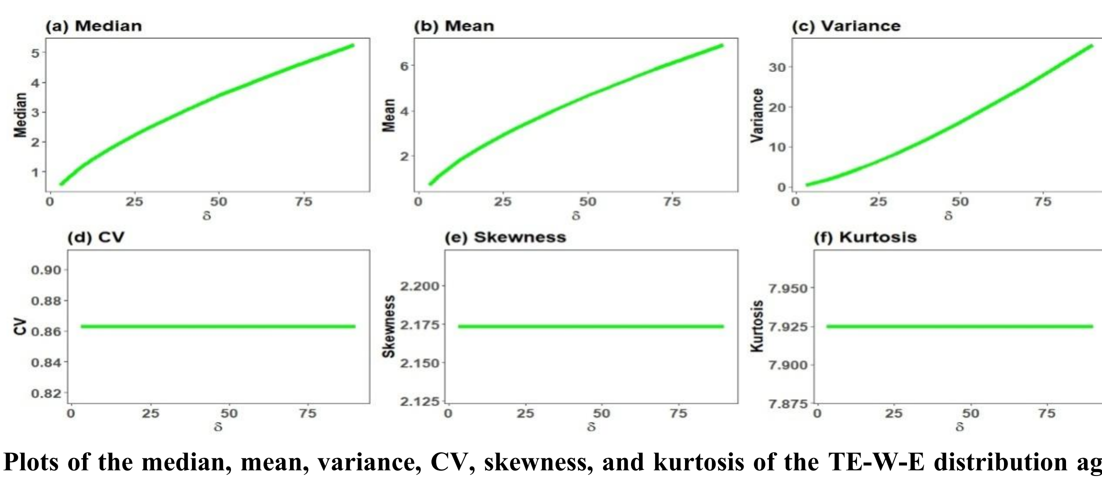

Figure 8: Plots of the median, mean, variance, CV, skewness, and kurtosis of the TE-W-E distribution againstδ for fixed$\boldsymbol{\omega} = \mathbf{0.4}, \boldsymbol{\alpha} = \mathbf{1.6}$and $\boldsymbol{\tau} = \mathbf{4}$

The plots in Figure 8 complements the empirical results in Table 4 which shows the effect of the compound parameter δ of the proposed TE-W-E distribution on the median, mean, variance, CV, skewness, and kurtosis. Figure 8a and Figure 8b show that both median and mean increase monotonically as δ increases. This implies that the scaling effect of δ shift the distribution toward larger median and mean values. Figure 8c show that the values of variance increase substantially as δ increases. It implies that the higher values of δ cause an increase in the dispersion of the distribution. Figure 8d show that the CV approximately constant for all values of δ. This indicates that while the absolute scale and spread increase, the relative variability, that is, spread relative to the mean keeps stable. Figure 8e shows that the coefficient of skewness remains constant at 2.1733 across all values of δ. This implies that the shape asymmetry of the TE-W-E distribution is constant under scale changes, that is, the δ does not affect the skewness of the proposed distribution. Figure 8f shows that the coefficient of kurtosis remains constant at 7.9247 for all values of δ and this implies that the distribution keeps its heavy-tailed and peaked characteristics, regardless of the scale parameter.

6. SIMULATION STUDY TO EVALUATE THE PERFORMANCE OF THE MAXIMUM LIKELIHOOD ESTIMATOR (MLE) ON THE ESTIMATION OF THE PARAMETERS OF TE-W-E DISTRIBUTION

Table 5: Average Values of the MLEs, Biases, Variances and MSEs of the TE-W-E Distribution for $\alpha = 2, \quad \omega = 0.8, \quad \tau = 4, \quad \delta = 8$

	MLE				
n	**Parameter**	**ML Mean**	$\lvert \boldsymbol{Bias} \rvert$	**Variance**	**MSE**
30	$\hat{\tau}$	9.0792	5.0792	0.0175	25.8158
	$\hat{\omega}$	-0.2358	1.0358	0.344	1.4169
	$\hat{\alpha}$	1.0126	0.9874	0.1351	1.1101
	$\hat{\delta}$	6.2205	1.7795	0.0301	3.1967
50	$\hat{\tau}$	8.0293	4.0293	0.0151	16.2504
	$\hat{\omega}$	-0.0219	0.8219	0.3076	0.9831
	$\hat{\alpha}$	1.9996	0.0004	0.1209	0.1209
	$\hat{\delta}$	6.8558	1.1442	0.0244	1.3336
100	$\hat{\tau}$	6.6645	2.6645	0.0131	7.1127
	$\hat{\omega}$	-0.1376	0.9376	0.2296	1.1087
	$\hat{\alpha}$	2.0630	0.063	0.0943	0.0983
	$\hat{\delta}$	7.5679	0.4321	0.0200	0.2067
500	$\hat{\tau}$	4.8108	0.8108	0.0116	0.6690
	$\hat{\omega}$	0.1744	0.6256	0.1185	0.5099
	α	2.0672	0.0672	0.0758	0.0803
	$\hat{\delta}$	8.3455	0.3455	0.0148	0.1342
1000	$\hat{\tau}$	4.4718	0.4718	0.0103	0.2329
	$\hat{\omega}$	0.1342	0.6658	0.1159	0.5592
	$\hat{\alpha}$	2.0228	0.0228	0.0524	0.0529
	$\hat{\delta}$	8.2263	0.2263	0.0117	0.0629
1500	$\hat{\tau}$	4.0281	0.0281	0.0093	0.0101
	$\hat{\omega}$	0.478	0.322	0.1146	0.2183
	$\hat{\alpha}$	2.0211	0.0211	0.0386	0.0390
	$\hat{\delta}$	8.1554	0.1554	0.0078	0.0319

Discussion of the Simulation Results

Monte Carlo simulation technique is used to generate samples of 30, 50, 100, 500, 1000 and 1500 for different combinations of parameters $(\tau, \omega, \alpha, \delta)$ from TE-W-E distribution. The simulation was repeated 10000 times for each sample size. The evaluation of ML estimator is based on its mean, the bias and the mean squared error (MSE) of the estimates. The estimates were obtained using optim function in R version 4.4.2. The result presented in Table 5 indicates the robust stability of the estimates from the MLE method. Notably, the estimates closely approximate the true values for the given sample sizes. Additionally, as the sample size (n) increases, the biases, variances and mean squared errors (MSEs) of τ, ω, α, and δ demonstrate a decreasing trend. Consequently, based on the findings of this simulation study, we affirm that the maximum likelihood estimation method is suitable for estimating the parameters of the TE-W-E model.

7. APPLICATION TO A REAL LIFE DATASET

The potentiality of TE-W-E distribution is illustrated in this section through the application of two real-life data sets. The TE-W-E distribution is compared with other competing distributions; Weibull (W) (Weibull, 1951), Weibull-Exponential (W-E) (Dubey, 1968), Weibull-Gamma (W-G) (Dubey, 1968), Kumaraswamy-Weibull (Kum-W) (Cordeiro et al., 2010), Transmuted Weibull Lomax (TWL) (Afify et al., 2015), Generalized Transmuted-Weibull (GT-W) (Nofal et al., 2017), Transmuted New Modified Weibull (TNMWD) (Vardhan & Balaswamy, 2016), Transmuted Lindley (TL) (Merovci, 2013a), Transmuted Exponential-Weibull (TE-W) (Mohammed & Ugwuowo, 2020) and New Exponential-Exponential (NE-E) (Ogunwale et al., 2022). The maximum likelihood estimates of the parameters of the distributions and its standard errors are obtained. Some assessment criteria are used to determine the best model out of the competing models. These criteria include Log-Likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC),Hannan-Quinn Information Criteria (HQIC) and Bayesian Information Criterion (BIC).

Furthermore, we equally compute other measures such as Anderson-Darling (A*), Cramér-Von Mises (W*) and Kolmogorov-Smirnov (D*) test statistic. The null hypothesis is that data follow a specified distribution. In general, model with the smallest value of the test statistics is considered to be the best fit among the competing models.

Dataset

The first dataset (Dataset I) represents the survival times of 121 patients with breast cancer obtained from a large hospital in the period from 1929 to 1938 by Lee(1992). This dataset had been used by Tahir et al. (2014), Al-kadim and Mahdi, (2018) and Jayakumar (2017). The dataset is also in the work by Mohammed and Ugwuowo (2021).

The second dataset (Dataset II) is on the survival times of Guinea Pigs injected with different doses of tubercle bacilli collected by Bjerkedal (1960). Recently, Okorie (2020) fitted the Transmuted Lindley (TL) distribution to this data.

Table 6: Descriptive statistics for Dataset I and Dataset II

	Sample size	Min.	Q1	Median	Mean	Q3	Max.	Variance	Skewness	Kurtosis
Dataset I	121	0.30	17.50	40.00	46.33	60.00	154.00	1244.464	1.0303	0.3461
Dataset II	72	12.00	54.75	70.00	99.82	112.75	376.00	6580.122	1.7590	2.4596

Table 6 shows that the dataset I and dataset II are positively skewed since their means, 46.33 and 99.82, are greater thantheir medians, 40.00 and 70.00, respectively. The coefficient of skewness for dataset I is 1.0303 with kurtosis coefficient of 0.3461. The coefficient of skewness for dataset II is 1.7590 with kurtosis coefficient of 2.4596. The results showed clearly that the datasets are asymmetric and positively skewed. The data were analyzed using R version 4.4.2 software. The skewness and kurtosis were obtained using fBasics package and the parameter estimates were obtained using optim function in R (R Core Team, 2024); and Table 7 and 8 present the computed maximum likelihood estimates and goodness-of-fit measures obtained.

Table 7: Estimated Parameters and Goodness-of-Fit for Dataset I (the survival times of 121 patients with breast cancer)

Model	ML Mean	-LL	AIC	CAIC	HQIC	BIC	W^*	A^*	D^*
TE-W-E	$\boldsymbol{\tau = 3.5763}$ $\boldsymbol{\omega = -0.8720}$ $\alpha =$ **1.0601** $\boldsymbol{\delta = 110.052}$	**582.5991**	**1173.1980**	**1165.27**	**1177.74**	**1171.4690**	**0.1418**	**0.8298**	**0.1049**
W-E	$\alpha = 1.0817$ $\delta = 31.7646$	611.2843	1226.569	1222.603	1228.84	1225.704	1.8454	9.9036	0.2035
W-G	$\beta = 1.0247$ $\gamma = 1.0834$ $\eta = 27.9630$	613.5541	1233.108	1227.159	1236.515	1231.8110	2.1582	11.2900	0.2210
TNMWD	$\alpha = 0.0085$ $\theta = 1.0819$ $\beta = 0.0052$ $\gamma = 2.1178$ $\lambda = -1.0047$ $\delta = 0.8543$	583.5162	1179.0320	1167.138	1185.845	1176.4390	0.2951	1.8174	0.1084
TWL	$a = 2.7063$ $b = 2.2201$ $\beta = 1.5680$ $\alpha = 0.1681$ $\lambda = -0.8720$	585.0436	1180.0870	1170.174	1185.765	1177.9260	0.1991	1.1344	0.1167
NE-E	$\lambda = 0.1469$	585.1277	1172.2550	1170.2720	1173.3910	1171.8230	0.4580	2.6884	0.1202
Kum-W	$a = 1.9198$ $b = 0.0852$ $c = 0.8959$ $\lambda = 0.3582$	587.3203	1182.6410	1174.71	1187.182	1180.9110	0.5016	2.9522	0.1240
GT-W	$\beta = 0.4028$ $\alpha = 0.4557$ $a = 9.8785$ $b = 6.0480$ $\lambda = -0.6228$	589.7075	1189.4150	1179.502	1195.092	1187.2540	0.1966	1.2939	0.1151
W	$\beta = 1.0332$ $\eta = 32.1508$	594.0937	1192.1870	1188.2210	1194.4580	1191.3230	13.4100	60.8220	0.6265
TL	$\theta = 1.0000$ $\lambda = 0.3193$	14803.64	29603.2800	29607.250	29601.010	29604.1400	38.263	70.1314	0.9590

Table 8: Estimated Parameters and Goodness-of-Fit for Dataset II (the survival times of Guinea Pigs injected with different doses of tubercle bacilli)

Model	ML Estimates	-LL	AIC	CAIC	HQIC	BIC	W^*	A^*	D^*
TE-W-E	$\boldsymbol{\tau = 1.3413}$ $\boldsymbol{\omega = -0.9942}$ $\boldsymbol{\alpha = 1.8299}$ $\boldsymbol{\delta = 1800.0}$	**389.8173**	**787.6347**	**779.7541**	**791.2601**	**785.4474**	**0.1009**	**0.5794**	**0.09221**
W-E	$\alpha = 1.8759$ $\delta = 3000.0$	394.2842	792.5683	788.6263	794.3810	791,4747	0.3241	1.8729	0.1613
W-G	$\beta = 1.3070$ $\gamma = 2.6417$ $\eta = 800.0027$	399.2953	804.5907	798.6789	807.3097	802.9502	0.6892	3.7841	0.21243
TL	$\alpha = 0.0159$ $\lambda = 0.0570$	393.2405	790.4809	786.5390	792.2937	789.3873	0.2597	1.4933	0.1084
TWL	$a = 1.6802$ $b = 3.9960$ $\beta = 1.9864$ $\alpha = 0.1531$ $\lambda = 0.3578$	393.7529	797.5058	787.6573	802.0376	794.7717	0.3102	1.7848	0.1464
TE-W	$\lambda = 0.5997$ $k = 1.5232$ $\gamma = 100.0046$ $\theta = 0.5974$	395.4364	798.8728	790.9922	802.4982	796.6855	0.3433	1.9626	0.1286
GT-W	$\beta = 1.3169$ $\alpha = 0.0100$ $a = 1.2709$ $b = 0.3464$ $\lambda = 0.3164$	395.8325	801.6651	791.8166	806.1968	798.9309	0.4072	2.1845	0.1446
NE-E	$\lambda = 0.1001$	403.4421	808.8843	806.9128	809.7906	808.3374	0.8060	4.4728	0.2116
W	$\beta = 1.3443$ $\delta = 100.7926$	397.6633	799.327	795.385	801.139	798.233	0.3512	2.372	0.1424

Table 7 and 8 show parameter maximum likelihood (ML) estimates to each one of the fitted distributions for the two datasets. The Table 7 and 8 also show goodness-of-fit statistics results from –LL, AIC, CAIC, HQIC, BIC,W^*, A^* and D^* for each distribution. The results show that the TE-W-E distribution has the smallest values of the goodness-of-fit statistics, –LL, AIC, CAIC, HQIC, BIC,W^*, A^* and D^*, when compared with other competing models (W, W-E, W-G, Kum-W, TWL, GT-W, TE-W, TNMW, TL and NE-E). This means that TE-W-E model has better performance and could be chosen as the best model compared to the other models used for fitting the dataset.

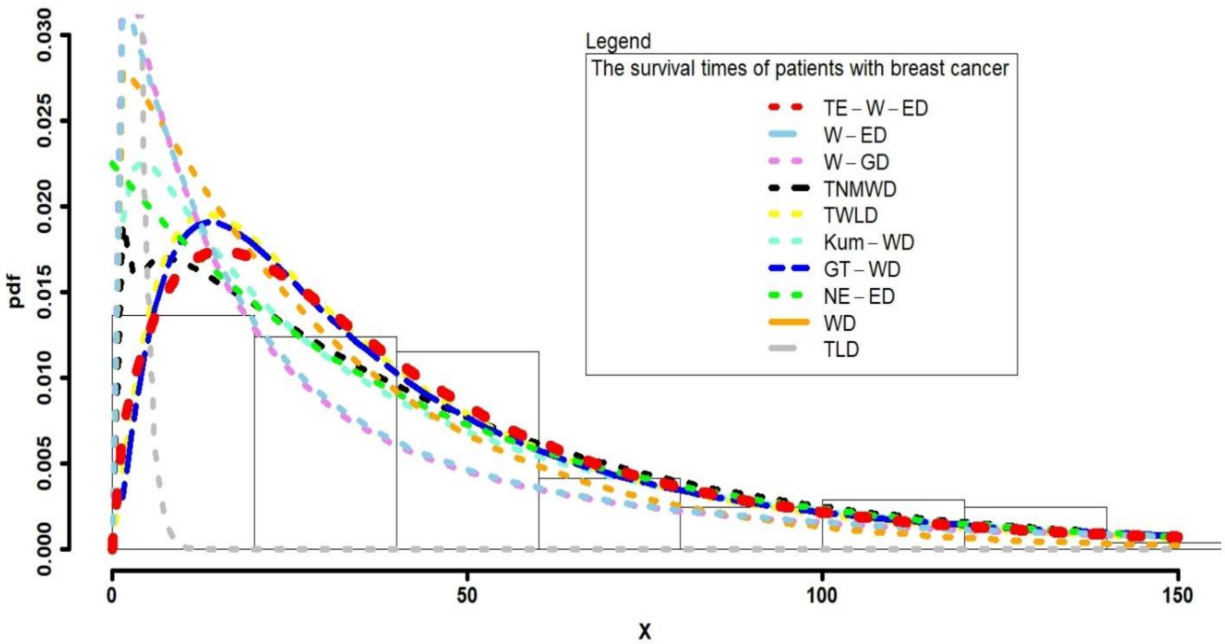

Figure 9: Plot of the estimated densities for the fitted models to dataset I (the survival times of patients with breast cancer)

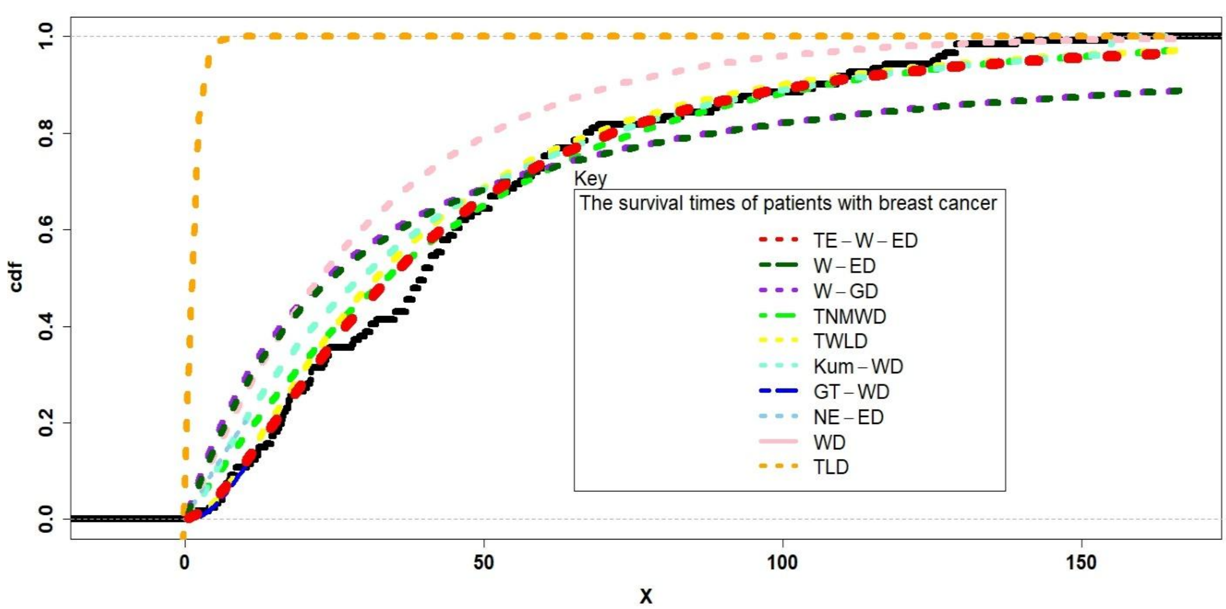

Figure 10:Plot of the cdf of the fitted models and empirical cdf to the dataset I (the survival times of patients with breast cancer)

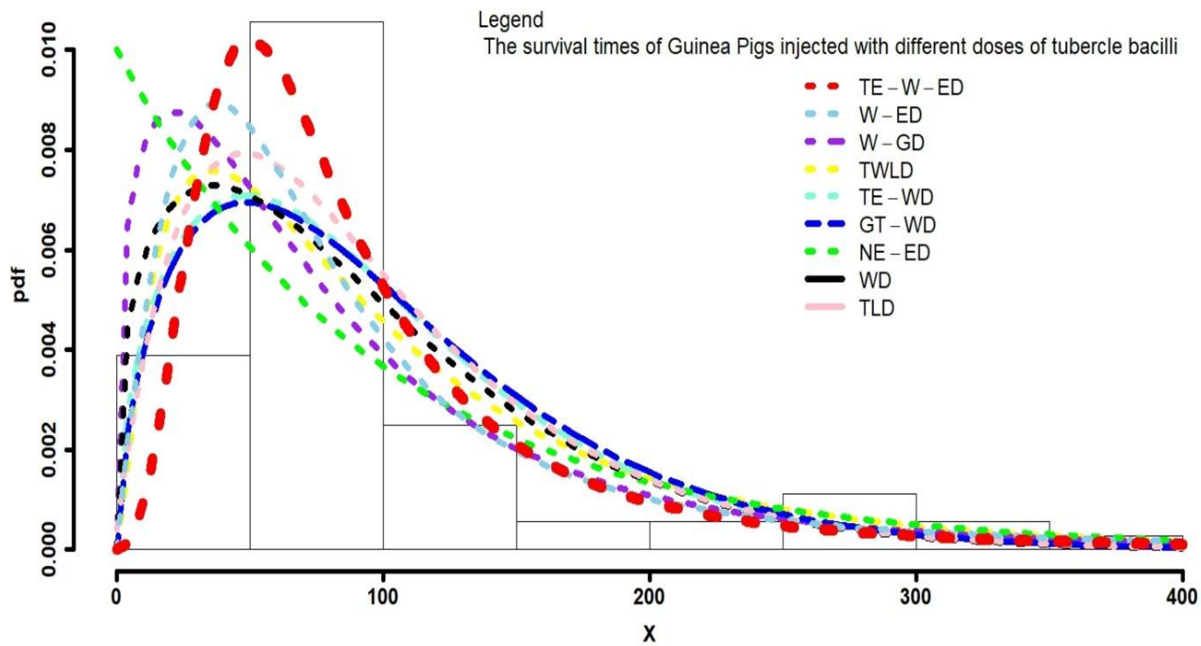

Figure 11: Plot of the estimated densities for the fitted models to dataset II (the survival times of Guinea Pigs injected with different doses of tubercle bacilli)

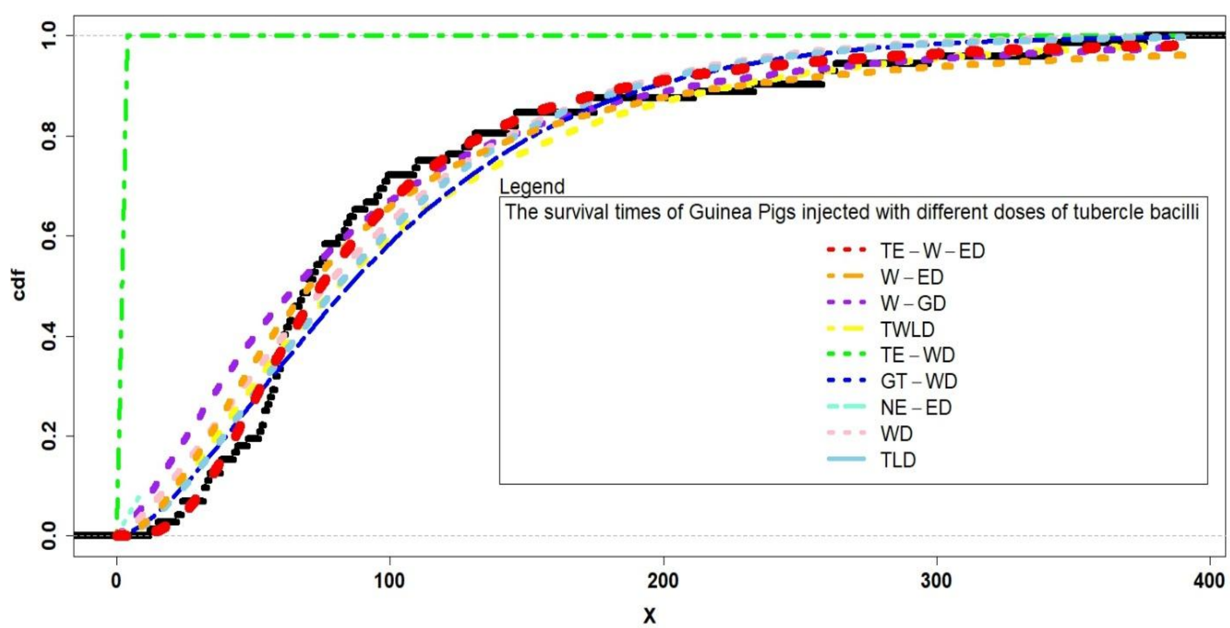

Figure 12:Plot of the cdf of the fitted models and empirical cdf to the dataset II (the survival times of Guinea Pigs injected with different doses of tubercle bacilli)

Discussion of results from the real life datasets

Figure 9 and 11 are the estimated density plot over histogram for the two considered real life datasets. Figure 10 andFigure 12 are the plot of cdf of the fitted models with empirical cdf. These plots demonstrated the performance of the proposed TE-W-E distribution on the considered real life datasets. As we can see from Figure 9 - 12, the TE-W-E model fitted well to the datasets than other fitted competing models and it is confirmed from the results of goodness-of-fit in Table 7 and 8. Therefore, TE-W-E model is the best model when compared with W, W-E, W-G, Kum-W, TWL, GT-W, TE-W, TNMW, TL and NE-E models on fitting to the two datasets which are positively skewed. These results reveal that the quadratic rank transmutation map by Shaw and Buckley (2007) is useful in increasing the skewness and flexibility of statistical distributions as showed in the research of Merovci and Puka (2014), Yousof et al. (2015), Oguntunde and Adejumo (2015), Owokolo et al. (2015), Mohammed and Ugwuowo (2020), and so on.

8. CONCLUSION

This study presents a new four-parameter distribution called the Transmuted Exponential-Weibull-Exponential (TE-W-E) distribution. The work provides a comprehensive investigation of its mathematical and statistical characteristics, deriving explicit formulas for its ordinary moments, moment-generating function, mean, variance, quantile function, survival function, hazard function, reversed hazard function, odd function, cumulative hazard function, Rényi entropy, and order statistics. Graphical plot of the PDF of the proposed TE-W-E model illustrates that the distribution is positively skewed and unimodal.

We conducted descriptive statistics to examine the effects of the four parameters of the proposed TE-W-E model on the distributional characteristics. The descriptive statistics presented include the minimum, first quartile, median, third quartile, maximum, mean, variance, standard deviation, CV, coefficient of skewness, and coefficient of kurtosis. The findings show that the shape parameterα, is the primary shape regulator of the proposed TE-W-E model. At small values of α, the distribution is highly skewed, heavy-tailed, and over-dispersed while at larger values of α, the distribution becomes more concentrated, less skewed, and exhibits lighter tails. This provides the proposed distribution the valuable flexibility for modeling various datasets; that is, making it capable of capturing both heavy-tailed and approximate symmetric datasets, depending on the chosenα value. The findings show that the transmuted parameter ω, is an additional powerful shape regulator that can fit the distribution to heavy-tailed, highly skewed and moderately skewed datasets. This flexibility improves the suitability of the TE-W-E model in modeling diverse real-world datasets, especially those having various degrees of skewness and kurtosis. The findings show that the scale parameter τ, plays an essential role in moderating the spread, asymmetry, and tail behavior of the proposed TE-W-E model. The smaller values of τ provide heavy-tailed, skewed, and highly dispersed distributions while the larger values of τ provide more symmetric, less dispersed, and lighter-tailed distributions. This indicates the flexibility of the TE-W-E model in modeling different datasets and it makes the TE-W-Emodel suitable for modeling heavy-tailed and approximate normal datasets, depending on the chosen τ values. The findings show that the compounding parameter δ, affects the location, median and mean, and dispersion, variance and standard deviation, of the proposed TE-W-E model. However, theCV, skewness, and kurtosis remains constant and it implies that δ only stretches or contracts the distribution without changing its shape characteristics. In general, the parameter δ, primarily controls the scale of the distribution, whereas the parameters ω, α, and τ are responsible for the shape dynamics; that is, the tail thickness and asymmetry of the TE-W-E distribution.

Parameter estimation is carried out through the Maximum Likelihood Estimation (MLE) method, with results demonstrating its effectiveness in accurately estimating the model's parameters. Two real-world datasets are employed to compare the TE-W-E model against other existing competing models. The results reveal that the TE-W-E achieves the lowest values for several goodness-of-fit measures which include Log-Likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC),Hannan-Quinn Information Criteria (HQIC), Bayesian Information Criterion (BIC), Anderson-Darling (A*), Cramér-Von Mises (W*) and Kolmogorov-Smirnov (D*) test statistic.; and this shows that the TE-W-E model outperforms other models in fitting the considered two real-world positively skewed data. The study also advances scientific knowledge by proposing survival and hazard functions applicable in diverse fields such as medicine, engineering, psychology, biological sciences, and epidemiology. Notably, the hazard function of the TE-W-E model exhibits a J-shaped or bathtub-shaped pattern, providing greater flexibility in modeling early failure risks, stable operational periods, and eventual wear-out phases; a versatility not offered by the Weibull or its baseline distribution, the Weibull-Exponential. Furthermore, the TE-W-E distribution is particularly well-suited for modeling time occurrences characterized by decreasing survival, increasing failure rates, or constant hazard patterns over time.

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