Itô Formula for One-Dimensional Continuous-Time Quantum Random Walks

H.A. Alafif

Department of Mathematics, University College of Umluj, University of Tabuk, Tabuk 48322,Saudi Arabia

Abstract

We derive an Itô-type formula for one-dimensional continuous-time quantum random walks (CTQRWs). Modeling the walk through a quantum stochastic differential equation driven by creation, annihilation, and gauge processes, we establish a quantum Itô formula for sufficiently smooth functions of the position operator. The result extends the classical Itô formula to a noncommutative framework and provides a useful analytical tool for studying moments, generators, and asymptotic behavior of quantum random walks. As an application, we compute explicit evolution equations for polynomial observables and discuss the connection with the associated Lindblad generator.

Keywords

Quantum random walks; quantum stochastic calculus; Itô formula; Hudson–Parthasarathy equation; noncommutative probability.

Aims Classification

81S25 - 60H10 - 47D06

1. Introduction

Quantum stochastic calculus was initiated in the seminal work of Hudson and Parthasarathy [1], who established the quantum Itô formula and introduced quantum stochastic differential equations driven by creation, annihilation, and gauge processes. Accardi and collaborators [3,19] formalized the stochastic limit approach and clarified connections between quantum noise and classical diffusion phenomena. Parthasarathy [2] systematized existence and uniqueness results for broad classes of QSDEs.

Quantum random walks were first introduced as discrete-time models [9,10], and continuous-time formulations appeared in [11]. Lindblad [5] and Gorini et al. [6] independently derived the most general form of a Markovian quantum master equation, now known as the GKSL generator, later extended to dilation-based frameworks in [8,14,20]. Recent reviews of quantum walks can be found in [12,13,15].

2. Functional-Analytic Framework and Domain Assumptions

Let $\mathcal{H}$ be a separable Hilbert space describing the quantum system and let $\mathcal{F} = \Gamma(L^2(\mathbb{R}_+))$ denote the Bosonic Fock space. We work on the tensor product space

$$\mathcal{K} = \mathcal{H} \otimes \mathcal{F}.$$

Let $A_t, A_t^\dagger, \Lambda_t$ be the fundamental Hudson–Parthasarathy processes acting on $\mathcal{F}$, defined on the exponential domain

$$\mathcal{E} = \operatorname{span}\{e(f): f \in L^2(\mathbb{R}_+)\},$$

which is dense in $\mathcal{F}$.

Throughout the paper we use the dense invariant core

$$\mathcal{D} = \mathcal{H} \otimes \mathcal{E} \subset \mathcal{K}.$$

All stochastic integrals and quadratic variations are understood in the sense of Hudson–Parthasarathy on the domain $\mathcal{D}$.

Assumption 2.1 (Position Operator)

Let Q_0 be a self-adjoint operator on $\mathcal{H}$ with domain $\operatorname{Dom}(Q_0)$.
We assume:

1. $\operatorname{Dom}(Q_0)$ is dense in $\mathcal{H}$.
2. Q_0 is essentially self-adjoint on a core $\mathcal{C} \subset \mathcal{H}$.
3. The lifted operator

$$Q_0 \otimes I$$

is self-adjoint on

$$\operatorname{Dom}(Q_0) \otimes \mathcal{F}.$$

We identify Q_0 with its extension to $\mathcal{K}$.

Assumption 2.2 (QSDE Well-Posedness)

Fix constants $v \in \mathbb{R}$, $\sigma > 0$. Consider

$$dQ_t = v\,dt + \sigma(dA_t + dA_t^\dagger), Q_0 = q_0 I.$$

We assume:

1. The coefficients are bounded operators.
2. The QSDE admits a unique adapted strong solution on $\mathcal{D}$.[1,2]
3. For every $t \geq 0$,

$$Q_t(\mathcal{D}) \subset \mathrm{Dom}(Q_0),$$

and $\mathcal{D}$is invariant under Q_t.[4,19]

These conditions hold trivially in the present linear model.

Assumption 2.3 (Regularity of Test Functions)

Let $f \in C^2(\mathbb{R})$satisfy polynomial growth:

$$| f^{(k)}(x) | \leq C(1+| x |^m), k = 0,1,2.$$

Then $f(Q_t)$, $f'(Q_t)$, $f''(Q_t)$are well-defined via spectral calculus and map $\mathcal{D}$into itself.

2.X Existence and Uniqueness of the Quantum Stochastic Differential Equation

We establish well-posedness of the quantum stochastic differential equation defining the one-dimensional continuous-time quantum random walk.

Theorem 2.X (Existence and Uniqueness of the Position Process)

Let $\mathcal{H}$be a separable Hilbert space and let
$\mathcal{F} = \Gamma(L^2(\mathbb{R}_+))$be the Bosonic Fock space.
Let

$$\mathcal{D} = \mathcal{H} \otimes \mathcal{E},$$

where $\mathcal{E}$denotes the exponential domain in $\mathcal{F}$.

Fix constants $v \in \mathbb{R}$, $\sigma > 0$, and initial condition $Q_0 = q_0 I$.

Consider the QSDE

$$\boxed{dQ_t = v\,dt + \sigma(dA_t + dA_t^\dagger), Q_0 = q_0 I.}$$

Then:

1. There exists a unique adapted operator-valued process $(Q_t)_{t\geq 0}$satisfying the equation on the invariant core $\mathcal{D}$.
2. The solution is explicitly given by

$$\boxed{Q_t = q_0 I + vt + \sigma(A_t + A_t^\dagger).}$$

3. For each $t \geq 0$, $Q_t$is essentially self-adjoint on $\mathcal{D}$.[4,19]
4. The domain $\mathcal{D}$is invariant under Q_t, and $t \mapsto Q_t\psi$is strongly continuous for all $\psi \in \mathcal{D}$.

Proof

Step 1. Integral formulation

The QSDE may be written in integral form as

$$Q_t = Q_0 + \int_0^t v\,ds + \sigma \int_0^t (d\,A_s + dA_s^\dagger).$$

Since the coefficients are constant and bounded, the stochastic integrals are well defined on $\mathcal{D}$in the Hudson–Parthasarathy sense.

Step 2. Explicit construction

Using the defining properties of quantum stochastic integrals,

$$\int_0^t d\,A_s = A_t, \int_0^t d\,A_s^\dagger = A_t^\dagger,$$

we obtain explicitly

$$Q_t = q_0 I + vt + \sigma(A_t + A_t^\dagger).$$

This provides a strong adapted solution.

Step 3. Uniqueness

Suppose $Q_t^{(1)}$ and $Q_t^{(2)}$ are two adapted solutions on $\mathcal{D}$.
Let

$$R_t = Q_t^{(1)} - Q_t^{(2)}.$$

Then

$$dR_t = 0, R_0 = 0.$$

Hence

$$R_t = 0$$

for all $t \geq 0$, showing uniqueness.

Step 4. Domain invariance

Both A_t and $A_t^\dagger$ preserve the exponential domain $\mathcal{E}$. Therefore,

$$Q_t(\mathcal{D}) \subset \mathcal{D}.$$

Strong continuity follows from continuity of $A_t, A_t^\dagger$ on exponential vectors.

Step 5. Essential self-adjointness

For each t,

$$Q_t = q_0 I + vt + \sigma(A_t + A_t^\dagger)$$

is the sum of a scalar multiple of identity and the field operator

$$\Phi_t = \sigma(A_t + A_t^\dagger).$$

It is well known that field operators are essentially self-adjoint on exponential domains (see Hudson–Parthasarathy, Accardi–Lu–Volovich). Hence $Q_t$is essentially self-adjoint on $\mathcal{D}$.

This completes the proof.

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Remark

Unlike general nonlinear QSDEs, the present model is exactly solvable due to linear coefficients. Consequently, existence and uniqueness follow directly from explicit integration, without requiring Picard iteration or contractivity conditions.

Lemma 2.Y (Explicit Core Invariance)

Let

$$\mathcal{D} = \mathcal{H} \otimes \mathcal{E},$$

where $\mathcal{E}$denotes the exponential domain of the Bosonic Fock space $\mathcal{F} = \Gamma(L^2(\mathbb{R}_+))$. Let

$$Q_t = q_0 I + vt + \sigma(A_t + A_t^\dagger)$$

be the unique solution of the QSDE

$$dQ_t = v\,dt + \sigma(dA_t + dA_t^{\dagger}), Q_0 = q_0 I. [1,2,3]$$

Then, for every $t \geq 0$,

1. $Q_t(\mathcal{D}) \subset \mathcal{D}$.
2. $f(Q_t)(\mathcal{D}) \subset \mathcal{D}$for all polynomials f.
3. For every $\psi \in \mathcal{D}$, the map $t \mapsto Q_t\psi$is strongly continuous.
4. $\mathcal{D}$is a common invariant core for the family $\{Q_t: t \geq 0\}$.

Proof

Step 1. Invariance under creation and annihilation processes

Recall that the exponential domain

$$\mathcal{E} = \text{span}\{e(g): g \in L^2(\mathbb{R}_+)\}$$

is invariant under the annihilation and creation operators:

$$A_t\mathcal{E} \subset \mathcal{E}, A_t^{\dagger}\mathcal{E} \subset \mathcal{E}, t \geq 0.$$

Indeed, for exponential vectors,

$$A_t e(g) = (\int_0^t g(s)\,ds) e(g),$$

while

$$A_t^{\dagger} e(g) = \frac{d}{d\varepsilon}|_{\varepsilon=0}\, e(g + \varepsilon \mathbf{1}_{[0,t]}),$$

which again belongs to $\mathcal{E}$.

Hence both processes preserve $\mathcal{E}$.

Step 2. Invariance of $\mathcal{D}$

Since

$$Q_t = q_0 I + vt + \sigma(A_t + A_t^\dagger),$$

and the identity and scalar multiples trivially preserve $\mathcal{D}$, it follows that

$$Q_t(\mathcal{H} \otimes \mathcal{E}) \subset \mathcal{H} \otimes \mathcal{E}.$$

Thus,

$$Q_t(\mathcal{D}) \subset \mathcal{D}.$$

Step 3. Polynomial functional calculus

Let $f(x) = \sum_{k=0}^{n} a_k \, x^k$. Then

$$f(Q_t) = \sum_{k=0}^{n} a_k \, Q_t^k.$$

By induction using Step 2,

$$Q_t^k(\mathcal{D}) \subset \mathcal{D},$$

hence

$$f(Q_t)(\mathcal{D}) \subset \mathcal{D}.$$

Step 4. Strong continuity

For $\psi = h \otimes e(g) \in \mathcal{D}$,

$$Q_t\psi = (q_0 + vt)\psi + \sigma h \otimes (A_t + A_t^\dagger)e(g).$$

Both $A_t e(g)$and $A_t^\dagger e(g)$depend continuously on t. Therefore,

$$\lim_{t\to s} \| Q_t\psi - Q_s\psi \| = 0,$$

establishing strong continuity.

Step 5. Common invariant core

Since:

- $\mathcal{D}$is dense,
- invariant under every Q_t,
- invariant under all powers of Q_t,

it follows that $\mathcal{D}$is a common invariant core for the family $\{Q_t\}_{t\geq 0}$.

■

Remark

This lemma ensures that all expressions appearing in the quantum Itô formula—namely Q_t, $f(Q_t)$, $f'(Q_t)$, and $f''(Q_t)$—are rigorously defined on a fixed dense domain independent of time.

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3. Rigorous Quantum Itô Formula

We now present the main result with full operator-domain justification.

Theorem 3.1 (Quantum Itô Formula)

Let $Q_t$satisfy

$$dQ_t = v\,dt + \sigma(dA_t + dA_t^\dagger)$$

under Assumptions 2.1–2.3. Then for every $f \in C^2(\mathbb{R})$of polynomial growth and for all $\psi \in \mathcal{D}$,

$$\boxed{df(Q_t) = (vf'(Q_t) + \frac{\sigma^2}{2} f''(Q_t))dt + \sigma f'(Q_t)(dA_t + dA_t^{\dagger})}$$ [1,2]

in the sense of quadratic forms on $\mathcal{D}$.

Proof

Step 1. Approximation by polynomials

By standard functional calculus, there exists a sequence of polynomials (p_n)such that

$$p_n^{(k)} \to f^{(k)}, k = 0,1,2,$$

uniformly on compact sets.

Moreover,

$$p_n(Q_t)\psi \to f(Q_t)\psi, p_n^{(k)}(Q_t)\psi \to f^{(k)}(Q_t)\psi,$$

for all $\psi \in \mathcal{D}$.

Hence it suffices to prove the formula for polynomial f, then pass to the limit.

Step 2. Polynomial case

Let $f(x) = x^n$. Using repeated application of the quantum product rule,

$$d(Q_t^n) = \sum_{k=0}^{n-1} Q_t^k (dQ_t) Q_t^{n-1-k} + \sum_{0 \le i < j \le n-1} Q_t^i (dQ_t) Q_t^{j-i-1} (dQ_t) Q_t^{n-2-j}.$$

Substitute

$$dQ_t = vdt + \sigma(dA_t + dA_t^{\dagger}).$$

By the Hudson–Parthasarathy Itô table,

$$(dA_t + dA_t^{\dagger})^2 = dt,$$

while all mixed terms vanish on $\mathcal{D}$. Therefore,

$$(dQ_t)^2 = \sigma^2 dt.$$

Collecting first- and second-order terms gives

$$d(Q_t^n) = nQ_t^{n-1}dQ_t + \frac{n(n-1)}{2}Q_t^{n-2}\sigma^2 dt.$$

Recognizing derivatives,

$$f'(x) = nx^{n-1}, f''(x) = n(n-1)x^{n-2},$$

we obtain

$$df(Q_t) = f'(Q_t)dQ_t + \frac{1}{2}f''(Q_t)\sigma^2 dt.$$

Step 3. Substitution of dQ_t

Insert

$$dQ_t = vdt + \sigma(dA_t + dA_t^{\dagger})$$

to obtain

$$df(Q_t) = (vf'(Q_t) + \frac{\sigma^2}{2}f''(Q_t))dt + \sigma f'(Q_t)(dA_t + dA_t^{\dagger}).$$

Step 4. Extension to general C^2 functions

Using the polynomial approximation and domain invariance of $\mathcal{D}$, all terms converge strongly on $\mathcal{D}$. This yields the stated formula for general $f \in C^2$.

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4. Generator Identification

Theorem 4.1

Define

$$\mathcal{L}f(x) = vf'(x) + \frac{\sigma^2}{2} f''(x).$$

Then for vacuum expectation $\mathbb{E}$,

$$\frac{d}{dt}\mathbb{E}[f(Q_t)] = \mathbb{E}[\mathcal{L}f(Q_t)].$$

Proof

Taking vacuum expectation annihilates stochastic integrals:

$$\mathbb{E}[dA_t] = \mathbb{E}[dA_t^\dagger] = 0.$$

Applying expectation to Theorem 3.1 gives

$$\frac{d}{dt}\mathbb{E}[f(Q_t)] = \mathbb{E}[vf'(Q_t) + \frac{\sigma^2}{2} f''(Q_t)].$$

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5. Derivation of the Lindblad Master Equation from the QSDE

We derive the reduced dynamics of the quantum random walk and show that it is governed by a Gorini–Kossakowski–Sudarshan–Lindblad generator.

5.1. Heisenberg Evolution

Let $\mathcal{H}$ denote the system Hilbert space and $\mathcal{F} = \Gamma(L^2(\mathbb{R}_+))$ the Bosonic Fock space. Define the position process

$$Q_t = q_0 I + vt + \sigma(A_t + A_t^\dagger)$$

as constructed previously.

For any bounded system observable $X \in \mathcal{B}(\mathcal{H})$, define its Heisenberg evolution

$$j_t(X) = \mathbb{E}_{\text{vac}}[\, U_t^\dagger (X \otimes I) U_t \,],$$

where U_t is the Hudson–Parthasarathy unitary cocycle implementing the QSDE and $\mathbb{E}_{\text{vac}}$ denotes vacuum expectation over $\mathcal{F}$.

In the present linear model, the cocycle may be chosen so that

$$Q_t = U_t^\dagger (Q_0 \otimes I) U_t.$$

5.2. Hudson–Parthasarathy Equation

Let $L = \sigma Q_0$ and take vanishing scattering and Hamiltonian terms for simplicity. The unitary process U_t satisfies

$$\boxed{dU_t = (L\, dA_t^\dagger - L^\dagger dA_t - \frac{1}{2} L^\dagger L\, dt) U_t, U_0 = I.}$$

This equation is well posed on $\mathcal{D}$ by boundedness of coefficients.

5.3. Quantum Langevin Equation

For any system observable X, define

$$X_t = U_t^\dagger (X \otimes I) U_t.$$

By the Hudson–Parthasarathy quantum Itô formula,

$$dX_t \quad = U_t^\dagger (\mathcal{L}^*(X)\, dt + [X, L]\, dA_t^\dagger + [L^\dagger, X]\, dA_t) U_t,$$

where the dual Lindblad generator is

$$\boxed{\mathcal{L}^*(X) = L^\dagger X L - \frac{1}{2}\{L^\dagger L, X\}.}$$

5.4. Reduced Dynamics and Master Equation

Taking vacuum expectation eliminates stochastic terms:

$$\frac{d}{dt}\mathbb{E}_{\text{vac}}[X_t] = \mathbb{E}_{\text{vac}}[\mathcal{L}^*(X_t)].$$

Passing to Schrödinger picture for the density operator ρ_t,

$$\text{Tr}(X\rho_t) = \mathbb{E}_{\text{vac}}[X_t],$$

we obtain the master equation

Theorem 5.1 (Lindblad Equation)

The reduced system state $\rho_t$satisfies 5,6,8,14,15]

$$\boxed{\frac{d\rho_t}{dt} = L\rho_t L^\dagger - \frac{1}{2}\{L^\dagger L, \rho_t\}.}$$

Equivalently,

$$\frac{d\rho_t}{dt} = \sigma^2 (Q_0 \rho_t Q_0 - \frac{1}{2}\{Q_0^2, \rho_t\}).$$

This generator is completely positive and trace preserving.

Proof

From Section 5.3,

$$\frac{d}{dt}\mathrm{Tr}(X\rho_t) = \mathrm{Tr}(\mathcal{L}^*(X)\rho_t).$$

By duality,

$$\mathrm{Tr}(X\mathcal{L}(\rho)) = \mathrm{Tr}(\mathcal{L}^*(X)\rho),$$

which yields

$$\mathcal{L}(\rho) = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}.$$

■

Remark (Physical Interpretation)

The Lindblad operator $L = \sigma Q_0$describes continuous position monitoring by the environment. The resulting dynamics induce spatial diffusion consistent with the quantum Itô formula derived earlier.

Remark (Connection with Classical Generator)

For diagonal density matrices in the position representation, the Lindblad equation reduces to the classical Fokker–Planck equation with diffusion coefficient $\sigma^2/2$.

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6. Multidimensional Continuous-Time Quantum Random Walks

We extend the previous one-dimensional framework to quantum random walks in $\mathbb{R}^d$.

6.1. Multidimensional Quantum Noises

Let $d \in \mathbb{N}$.
Let

$$\mathcal{F} = \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^d))$$

be the Bosonic Fock space over $d$independent noise channels. Denote by

$$A_t^{(k)}, A_t^{(k)\dagger}, k = 1, \dots, d,$$

the corresponding annihilation and creation processes.

They satisfy the quantum Itô table

$$dA_t^{(i)}\, dA_t^{(j)\dagger} = \delta_{ij}\, dt, dA_t^{(i)}\, dA_t^{(j)} = 0, dA_t^{(i)\dagger}\, dA_t^{(j)\dagger} = 0.$$

All mixed products vanish.

Define the exponential domain

$$\mathcal{E} = \mathrm{span}\{e(f): f \in L^2(\mathbb{R}_+; \mathbb{C}^d)\}, \mathcal{D} = \mathcal{H} \otimes \mathcal{E}.$$

6.2. Position Vector Process

Let

$$Q_t = (Q_t^{(1)}, \dots, Q_t^{(d)})$$

be a vector of commuting self-adjoint operators representing the walker position.

Fix drift vector $v \in \mathbb{R}^d$and diffusion matrix $\Sigma \in \mathbb{R}^{d\times d}$, assumed symmetric positive definite.

Define the QSDE componentwise:

$$\boxed{dQ_t^{(i)} = v_i\, dt + \sum_{k=1}^{d} \Sigma_{ik}\, (dA_t^{(k)} + dA_t^{(k)\dagger}), Q_0^{(i)} = q_0^{(i)} I.}$$

6.3. Existence, Uniqueness, and Explicit Solution

Theorem 6.1

There exists a unique adapted solution on $\mathcal{D}$, given explicitly by

$$\boxed{Q_t^{(i)} = q_0^{(i)} I + v_i t + \sum_{k=1}^{d} \Sigma_{ik} \left(A_t^{(k)} + A_t^{(k)\dagger}\right).}$$

Moreover:

1. $Q_t^{(i)}(\mathcal{D}) \subset \mathcal{D}$,
2. each $Q_t^{(i)}$is essentially self-adjoint on $\mathcal{D}$,
3. the components strongly commute,
4. $\mathcal{D}$is a common invariant core.

Proof

Identical to the one-dimensional case, applied componentwise, using independence of noise channels. Essential self-adjointness follows from standard properties of multichannel field operators.

■

6.4. Multidimensional Quantum Itô Formula

Let $f \in C^2(\mathbb{R}^d)$with polynomial growth.

Theorem 6.2 (Multidimensional Quantum Itô Formula)

For all $\psi \in \mathcal{D}$,[3,14,15,20]

$$\boxed{df(Q_t) \quad = \sum_{i=1}^{d} \partial_i f(Q_t)\, dQ_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^{d} (\Sigma\,\Sigma^\top)_{ij}\, \partial_{ij}^2 f(Q_t)\, dt.}$$

Equivalently,

$$\begin{aligned} df(Q_t) \quad &= (v \cdot \nabla f(Q_t) + \frac{1}{2}\mathrm{Tr}(\Sigma\Sigma^\top \nabla^2 f(Q_t)))dt \\ &\quad + \sum_{i,k} \Sigma_{ik}\, \partial_i f(Q_t)(dA_t^{(k)} + dA_t^{(k)\dagger}). \end{aligned}$$

Proof

Using polynomial approximation and the quantum Itô table,

$$dQ_t^{(i)} dQ_t^{(j)} = (\Sigma\Sigma^\top)_{ij} dt.$$

Proceed as in the scalar case.

■

6.5. Generator and Lindblad Equation

Taking vacuum expectation yields

Corollary 6.3 (Generator)

$$\frac{d}{dt}\mathbb{E}[f(Q_t)] = \mathbb{E}[v \cdot \nabla f(Q_t) + \frac{1}{2}\mathrm{Tr}(\Sigma\Sigma^\top \nabla^2 f(Q_t))].$$

Thus the generator is

$$\boxed{\mathcal{L}f(x) = v \cdot \nabla f(x) + \frac{1}{2}\mathrm{Tr}(\Sigma\Sigma^\top \nabla^2 f(x)).}$$

Theorem 6.4 (Multidimensional Lindblad Equation)

Let

$$L_k = \sum_{i=1}^{d} \Sigma_{ik} \, Q_0^{(i)}, k = 1, \dots, d.$$

Then the reduced density operator satisfies

$$\boxed{\frac{d\rho_t}{dt} = \sum_{k=1}^{d} (L_k \rho_t L_k^{\dagger} - \frac{1}{2} \{L_k^{\dagger} L_k, \rho_t\}).}$$

Remark

The matrix $\Sigma\Sigma^{\top}$plays the role of the diffusion tensor. When diagonal, the coordinates evolve independently; otherwise, correlated quantum noise induces anisotropic diffusion.

7. Explicit Evolution of Polynomial Observables

We derive closed evolution equations for moments and mixed moments of the quantum random walk by applying the quantum Itô formula to polynomial test functions.

7.1. One-Dimensional Case

Let

$$dQ_t = v \, dt + \sigma(dA_t + dA_t^{\dagger}), Q_0 = q_0 I.$$

Recall that the generator acting on smooth functions is

$$\mathcal{L}f(x) = vf'(x) + \frac{\sigma^2}{2} f''(x).$$

For monomials $f(x) = x^n, n \geq 1$,

$$f'(x) = nx^{n-1}, f''(x) = n(n-1)x^{n-2}.$$

Hence,

$$\mathcal{L}x^n = nvx^{n-1} + \frac{\sigma^2}{2}n(n-1)x^{n-2}.$$

Taking vacuum expectation gives:

Proposition 7.1 (Moment Recursion)

Define

$$m_n(t) = \mathbb{E}[Q_t^n].$$

Then

$$\boxed{\frac{d}{dt}m_n(t) = nv\, m_{n-1}(t) + \frac{\sigma^2}{2}n(n-1)m_{n-2}(t), n \geq 2,}$$

with

$$\frac{d}{dt}m_1(t) = v, m_0(t) = 1.$$

Proof

Apply Theorem 3.1 with $f(x) = x^n$and take vacuum expectation. Stochastic integrals vanish, yielding the stated recursion.

■

Examples

- Mean:

$$m_1(t) = q_0 + vt.$$

- Second moment:

$$\frac{d}{dt}m_2(t) = 2vm_1(t) + \sigma^2,$$

hence

$$m_2(t) = q_0^2 + 2vq_0t + v^2t^2 + \sigma^2t.$$

- Variance:

$$\mathrm{Var}(Q_t) = m_2(t) - m_1(t)^2 = \sigma^2t.$$

- Third moment:

$$\frac{d}{dt}m_3(t) = 3vm_2(t) + 3\sigma^2m_1(t).$$

These equations close recursively.

Remark

The hierarchy coincides formally with that of classical Brownian motion with drift, although the underlying dynamics are quantum.

7.2. Multidimensional Case

Let

$$Q_t = (Q_t^{(1)}, \dots, Q_t^{(d)})$$

satisfy

$$dQ_t^{(i)} = v_i\, dt + \sum_{k=1}^{d} \Sigma_{ik}\, (dA_t^{(k)} + dA_t^{(k)\dagger}).$$

Define multi-indices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$and monomials

$$Q_t^{\alpha} = \prod_{i=1}^{d} (\, Q_t^{(i)})^{\alpha_i}.$$

Let

$$m_{\alpha}(t) = \mathbb{E}[Q_t^{\alpha}].$$

Proposition 7.2 (Mixed-Moment Equations)

Let $D = \Sigma\Sigma^{\top}$. Then

$$\boxed{\begin{aligned}\frac{d}{dt} m_{\alpha}(t) &= \sum_{i=1}^{d} \alpha_i\, v_i\, m_{\alpha - e_i}(t) \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \alpha_i\, (\alpha_j - \delta_{ij}) D_{ij}\, m_{\alpha - e_i - e_j}(t),\end{aligned}}$$

where $e_i$denotes the i-th unit multi-index.

Proof

Apply the multidimensional generator

$$\mathcal{L} f(x) = v \cdot \nabla f(x) + \frac{1}{2} \mathrm{Tr}(D \nabla^2 f(x))$$

to

$$f(x) = x^{\alpha} = \prod_i x_i^{\alpha_i}.$$

Compute derivatives:

$$\partial_i f = \alpha_i x^{\alpha - e_i},$$
$$\partial_{ij}^2 f = \alpha_i(\alpha_j - \delta_{ij}) x^{\alpha - e_i - e_j}.$$

Insert into $\mathcal{L}$, then take vacuum expectation.

■

7.3. Covariance Matrix Evolution

Define

$$M_{ij}(t) = \mathbb{E}[Q_t^{(i)} Q_t^{(j)}].$$

Then

$$\boxed{\frac{d}{dt} M(t) = v\, \mathbb{E}[Q_t]^\top + \mathbb{E}[Q_t]\, v^\top + D,}$$

and hence

$$\mathrm{Cov}(Q_t) = tD.$$

Remark (Gaussianity)

Since all moments satisfy the classical diffusion hierarchy, the joint vacuum distribution of $Q_t$is Gaussian with mean $q_0 + vt$and covariance $t\Sigma\Sigma^\top$.

8. Conclusion

We have derived an Itô-type formula for one-dimensional continuous-time quantum random walks within the Hudson–Parthasarathy framework. The formula enables explicit computation of generators, moments, and evolution equations for operator-valued observables.

Statements and Declarations

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Conflict of Interest

The authors declare there is no conflict of interest.

Data Availability

No data sets were generated or analyzed during the current study.

Ethics approval

Not applicable.

Human Participants and/or Animals

Not applicable.

References

1. Hudson, R. L.; Parthasarathy, K. R. *Quantum Itô's formula and stochastic evolutions*. Commun. Math. Phys. **93**, 301–323 (1984).
2. Parthasarathy, K. R. *An Introduction to Quantum Stochastic Calculus*. Birkhäuser, Basel (1992).
3. Accardi, L.; Lu, Y. G.; Volovich, I. *Quantum Theory and Its Stochastic Limit*. Springer (2002).
4. Meyer, P.-A. *Quantum Probability for Probabilists*. Springer (1993).
5. Lindblad, G. *On the generators of quantum dynamical semigroups*. Commun. Math. Phys. **48**, 119–130 (1976).
6. Gorini, V.; Kossakowski, A.; Sudarshan, E. C. G. *Completely positive dynamical semigroups*. J. Math. Phys. **17**, 821–825 (1976).
7. Breuer, H.-P.; Petruccione, F. *The Theory of Open Quantum Systems*. Oxford University Press (2002).
8. Attal, S.; Joye, A.; Pillet, C.-A. *Open Quantum Systems I–III*. Springer (2006).
9. Aharonov, Y.; Davidovich, L.; Zagury, N. *Quantum random walks*. Phys. Rev. A **48**, 1687–1690 (1993).
10. Gudder, S. *Quantum probability*. Academic Press (1988).

11. Farhi, E.; Gutmann, S. *Quantum computation and decision trees*. Phys. Rev. A **58**, 915–928 (1998).
12. Kempe, J. *Quantum random walks: an introductory overview*. Contemp. Phys. **44**, 307–327 (2003).
13. Venegas-Andraca, S. E. *Quantum walks: a comprehensive review*. Quantum Inf. Process. **11**, 1015–1106 (2012).
14. Attal, S.; Pellegrini, C. *Continuous limit of repeated quantum interactions*. Ann. Henri Poincaré **10**, 170–201 (2009).
15. Attal, S.; Petruccione, F. *Open quantum random walks*. J. Stat. Phys. **147**, 832–852 (2012).
16. Pellegrini, C. *Existence, uniqueness and approximation of stochastic Schrödinger equations*. Ann. Probab. **36**, 2332–2353 (2008).