Transmuted Exponential-New Weibull Pareto Distribution: A Flexible Probability Model for Positively Skewed and Heavy-tailed Lifetime Data.

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Abstract- Many lifetime datasets encountered in reliability engineering, survival analysis, and other applied sciences exhibit complex characteristics such as skewness, heavy tails, elongation, and varying hazard rate structures, which render classical probability distributions inadequate for accurate modelling. To address this limitation, this study proposes a new flexible lifetime model, termed the Transmuted Exponential–New Weibull Pareto (TE–NWP) distribution, obtained by applying the Transmuted Exponential-G (TE–G) family to the New Weibull Pareto distribution. The proposed model possesses a flexible five-parameter structure suitable for analysing positive data and is capable of capturing skewed and heavy-tailed lifetime behaviours. In addition, the model exhibits a bathtub-shaped hazard rate function, providing greater flexibility in modelling datasets characterized by elongation and asymmetry compared to several existing distributions. Several important statistical properties of the distribution are derived, including the moments, moment generating function, quantile function, survival function, and hazard function. The model parameters are estimated using the maximum likelihood estimation (MLE) method. A Monte Carlo simulation study is conducted to evaluate the performance of the estimators in terms of bias, variance, and mean squared error under varying sample sizes. Finally, the applicability of the proposed distribution is demonstrated using two real-life datasets. The results indicate that the TE–NWP distribution provides a superior fit compared to several competing models based on information criteria and goodness-of-fit.

Keywords: Transmuted Exponential-G family, New Weibull Pareto distribution, Lifetime data, Maximum likelihood estimation, Heavy-tailed and skewed data

1. Introduction

In disciplines such as reliability studies, survival analysis, computer science, and the social sciences, the flexibility of probability distributions is essential for both theoretical development and practical applications, as it determines the ability to accurately model diverse data patterns that are observed in real-life data. (Mahmoud et al., 2017).

The existing distributions, such as the Exponential, Weibull, Gompertz, Pareto, and Normal distributions, often struggle to capture complex data characteristics, such as heavy tails, skewness, or bathtub-shaped hazard rates. This limitation has motivated the development of modified and generalized distributions that incorporate additional parameters, thereby improving their adaptability to complex real-life data (Alzaatreh et al., 2013).

Several efforts have been made to design more adaptable distributions. Early contributions include parameter estimation of modified Weibull distributions (Zaindin & Sarhan, 2009), the transmuted Weibull distribution (Aryal & Tsokos, 2011), Exponential Pareto distribution (Al-Kadim, & Boshi, 2013), Further extensions were later introduced, such as the Weibull-Lomax distribution (Cordeiro et al., 2015), mixtures of Burr XII and Weibull (Daniyal & Aleem, 2014), the New Weibull-Pareto Distribution (Nasiru, & Luguterah, 2015), the Exponentiated Weibull-Pareto Distribution(Afify et al., 2016), Weibull–Rayleigh (Ahmad et al., 2017), and the Lindley Weibull (Cordeiro et al., 2018). Other notable contributions are the Fréchet–Weibull distribution with applications to earthquake data (Teamah et al., 2020), the Transmuted Generalized Odd Generalized Exponential-G family (Hesham Reyad et al., 2021), The Exponentiated Transmuted Kumaraswamy distribution (Joseph & Ravindran, 2023), Transmuted new weighted exponential distribution (Abdullahi et al., 2024), Transmuted Exponential-Compound Weibull Distribution (Eze & Yahya, 2025).

A particularly influential advancement in this direction is the introduction of the Transmuted-G family by Shaw & Buckley (2007), which incorporates a transmutation parameter to increase the flexibility of baseline models. Their work inspired a surge of research on transmuted models, demonstrating superior performance in modelling skewed and heavy-tailed data (Nwezza & Ugwuowo, 2021). Building on this foundation, Mohammed & Ugwuowo (2020) proposed the Transmuted Exponential-G (TE–G) which introduces both transmutation and scale parameters. When applied to the New Weibull Pareto distribution, the TE–G produced the Transmuted Exponential–New Weibull Pareto (TE–NWP) distribution, which is effective for positively skewed data. Motivated by this development, the present study applies the TE–G to the New Weibull Pareto (NWP) distribution, yielding the Transmuted Exponential–New Weibull Pareto (TE–NWP) distribution. This new model introduces additional flexibility, making it well-suited for heavy-tailed and positively skewed lifetime data, and addressing the shortcomings of existing distributions.

The aim of this research is to propose, explore and evaluate the TE-NWP distribution and its performance using simulated and real-life datasets. The remainder of this paper is structured as follows: Section 2 introduces the CDF and PDF of the proposed model along with several statistical properties. Section 3 discuses parameter estimation using maximum likelihood method and also discusses the simulation results, Section 4 illustrates real data applications, and Section 5 concludes with remarks and directions for future research.

2. Material and Methods

For a continuous random variable X, the cumulative distribution function (CDF) and probability density function (PDF) of the TE–G family of distributions, as defined by Mohammed and Ugwuowo (2020), are respectively expressed as follows:

$$F(x;\tau,\theta,\psi) = \left(1-\left(1-H(x;\psi)\right)^{\tau}\right)\left(1+\theta\left(1-H(x;\psi)\right)^{\tau}\right) \quad (1)$$

and

$$f(x;\tau,\theta,\psi) = \frac{\mathrm{h}(\mathrm{x};\psi)}{1-\mathrm{H}(\mathrm{x};\psi)}\tau\{1-\mathrm{H}(\mathrm{x};\psi)\}^{\tau}(1-\theta+2\theta\{1-\mathrm{H}(\mathrm{x};\psi)\}^{\tau}) \quad (2)$$

Where;

$\mathrm{H}(\mathrm{x};\psi)$ is the cdf of any continuous distribution which depends on a parameter vector ψ

$\mathrm{h}(\mathrm{x};\psi)$ is the pdf of any continuous distribution which depends on a parameter vector ψ

τ is the additional scale parameter; $\tau > 0$ and θ is the transmuted parameter; $-1 \leq \theta \leq 1$

2.1 Derivation Of the Transmuted Exponential–New Weibull Pareto (TE–NWP) Distribution

Let the density function of the New Weibull Pareto (NWP) distribution be given by

$\mathrm{h}(\mathrm{x};\psi) = \frac{\beta\sigma}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1} exp - \left(\sigma\left(\frac{x}{\gamma}\right)^{\beta}\right)$ and its distribution function given by $H(x;\psi) = 1 - exp - \left(\sigma\left(\frac{x}{\gamma}\right)^{\beta}\right)$ where $\sigma > 0$ and $\gamma > 0$ are the scale parameters and $\beta > 0$ as the shape parameter (see Nasiru & Lugutera 2015). By substituting the $\mathrm{h}(\mathrm{x};\psi)$ and $H(x;\psi)$ in (1) and (2), the cumulative distribution function (CDF) and probability density function (PDF) of TE-NWP distribution are respectively given as:

$$F(x;\tau,\theta,\gamma,\sigma,\beta) = \left\{1 - exp\left(-\tau\sigma\left(\frac{x}{\gamma}\right)^{\beta}\right)\right\}\left(1 + \theta\left\{exp\left(-\tau\sigma\left(\frac{x}{\gamma}\right)^{\beta}\right)\right\}\right) \tag{3}$$

and

$$f(\mathrm{x};\tau,\theta,\gamma,\sigma,\beta) = \frac{\beta\sigma\tau}{\gamma}\left(\frac{\mathrm{x}}{\gamma}\right)^{\beta-1}(1-\theta)\exp\left(-\tau\,\sigma\left(\frac{\mathrm{x}}{\gamma}\right)^{\beta}\right) + \frac{2\theta\tau\beta\sigma}{\gamma}\left(\frac{\mathrm{x}}{\gamma}\right)^{\beta-1}\exp\left(-2\tau\,\sigma\left(\frac{\mathrm{x}}{\gamma}\right)^{\beta}\right) \tag{4}$$

where, $\gamma,\sigma,\beta,\tau > 0$ and $-1 \leq \theta \leq 1$

2.2. **Model Validity Check**

The validity of the TE–NWP distribution is established by showing that its pdf integrates to unity over $(0,\infty)$. Using the Gamma integral identity to evaluate the component integrals, it follows that

$$\int_0^{\infty} f(x;\tau,\theta,\gamma,\sigma,\beta)\,dx = \frac{\beta\sigma\tau(1-\theta)}{\gamma^{\beta}} \times \frac{\gamma^{\beta}}{\beta\tau\,\sigma} + \frac{2\theta\tau\beta\sigma}{\gamma^{\beta}} \times \frac{\gamma^{\beta}}{2\beta\tau\,\sigma} = 1,$$

hence confirming it is a valid probability density function.

2.3 Graphical Illustration of Density Function and the Distribution Function of the TE-NWP Distribution

The plots of probability density function (PDF) and cumulative distribution function (CDF) of TE-NWP distribution are respectively shown in figure 1 and 2 for selected values of the parameters.

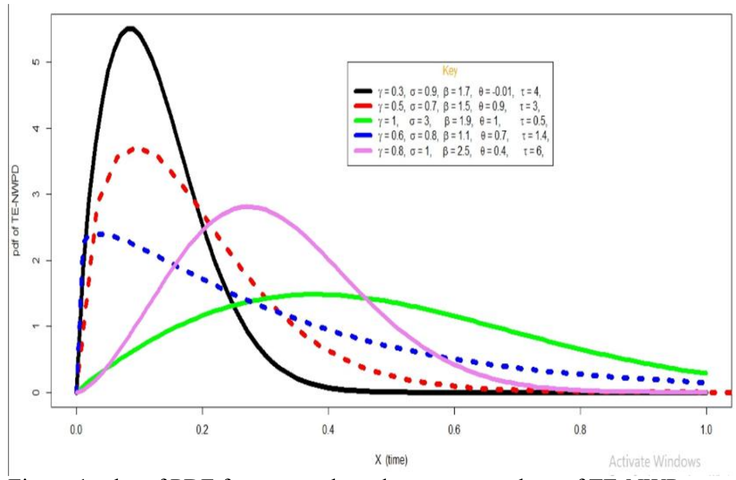

Figure 1: plot of PDF for some selected parameter values of TE-NWP distribution

Figure 1 presented multiple shapes of the Transmuted Exponential-New Weibull Pareto Distribution (TE-NWPD) under different parameter values. These shapes reveal that the TE-NWPD is consistently skewed to the right, making it a practical choice for modelling positively skewed datasets. This characteristic is especially beneficial when handling data where higher values occur less frequently but remain significant, such as in income distributions, failure rates, and life span studies.

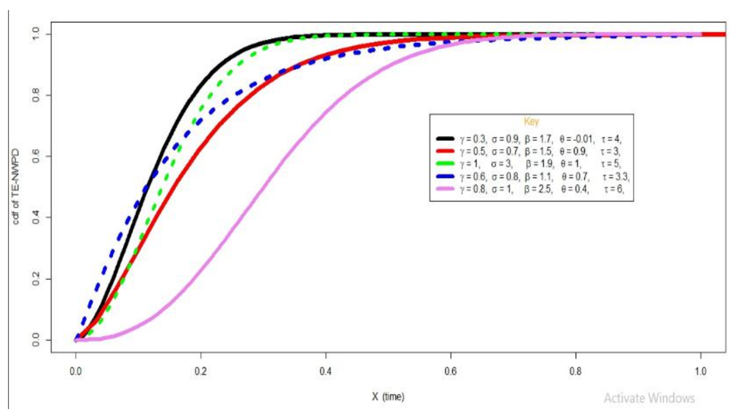

Figure 2: plot of CDF for some selected values of TE-NWP distribution

Figure 2 demonstrates that as the values of the random variable X increases, the lines progressively converge toward 1. Conversely, as X approaches zero, the lines similarly converge toward zero. This pattern confirms that the Transmuted Exponential-

New Weibull Pareto (TE-NWP) Distribution behaves consistently within the expected range, validating it as a legitimate cumulative distribution function.

2.4 Statistical and Mathematical Properties of TE-NWP Distribution

In this section, we discuss certain properties of the TE-NWP distribution, which are highlighted as follows:

2.4.1 Moment of TE-NWP distribution

Let X be a random variable with PDF $f(x)$, following TE-NWP distribution as defined in (4). The rth moment about the origin is expressed as:

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x;\tau,\theta,\gamma,\sigma,\beta)dx$$

substituting the pdf and simplifying using the Gamma integral identity yields

$$\mu_r' = \frac{\gamma^r\Gamma\left(1+\frac{r}{\beta}\right)}{(\tau\,\sigma)^{\frac{r}{\beta}}}\left(1-\theta+\frac{\theta}{2^{\frac{r}{\beta}}}\right) \tag{5}$$

Substituting $r = 1,2,3,4$ in Equation (5), we obtain

$$\mu_1' = \frac{\gamma\Gamma\left(1+\frac{1}{\beta}\right)}{(\tau\,\sigma)^{\frac{1}{\beta}}}\left(1-\theta+\frac{\theta}{2^{\frac{1}{\beta}}}\right), \qquad \mu_2' = \frac{\gamma^2\Gamma\left(1+\frac{2}{\beta}\right)}{(\tau\,\sigma)^{\frac{2}{\beta}}}\left(1-\theta+\frac{\theta}{2^{\frac{2}{\beta}}}\right),$$

$$\mu_3' = \frac{\gamma^3\Gamma\left(1+\frac{3}{\beta}\right)}{(\tau\,\sigma)^{\frac{3}{\beta}}}\left(1-\theta+\frac{\theta}{2^{\frac{3}{\beta}}}\right), \qquad \mu_4' = \frac{\gamma^4\Gamma\left(1+\frac{4}{\beta}\right)}{(\tau\,\sigma)^{\frac{4}{\beta}}}\left(1-\theta+\frac{\theta}{2^{\frac{4}{\beta}}}\right)$$

2.4.2 rth moment about the mean (rth central moments) of TE-NWP distribution

Central moments are essential in describing distributional properties as they enable the computation of variance, skewness and kurtosis among other key descriptors of the shape of a distribution. The rth central moment is mathematically defined as follows:

$$E(x-\mu)^r = \int_0^\infty (x-\mu)^r f(x;\tau,\theta,\gamma,\sigma,\beta)\,dx$$

By applying binomial expansion,

$$E(x-\mu)^r = \frac{\beta\sigma\tau}{\gamma^\beta}\sum_{n=0}^{\infty}(-1)^n\binom{r}{n}\mu^n\left(\frac{(1-\theta)}{\beta}\left(\frac{\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+r-n)}{\beta}}\Gamma\left(\frac{\beta+r-n}{\beta}\right)\right) + \frac{\beta\sigma\tau}{\gamma^\beta}\sum_{n=0}^{\infty}(-1)^n\binom{r}{n}\mu^n\left(\frac{2\theta}{\beta}\left(\frac{2\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+r-n)}{\beta}}\Gamma\left(\frac{\beta+r-n}{\beta}\right)\right) \tag{6}$$

On simplifying Equation (6), we obtain

$$E(x-\mu)^r = \sum_{n=0}^{\infty}(-1)^n\binom{r}{n}\mu^n\left(\frac{\sigma\tau(1-\theta)}{\gamma^\beta}\left(\frac{\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+r-n)}{\beta}} + \frac{2\theta\sigma\tau}{\gamma^\beta}\left(\frac{2\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+r-n)}{\beta}}\right)\Gamma\left(\frac{\beta+r-n}{\beta}\right) \tag{7}$$

from Equation (7), the corresponding rth moment about the mean for $r = 1,2,3, and\ 4$ are given respectively as:

$$E(x-\mu) = \sum_{n=0}^{\infty}(-1)^n\binom{1}{n}\mu^n\left(\frac{\sigma\tau(1-\theta)}{\gamma^\beta}\left(\frac{\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+1-n)}{\beta}} + \frac{2\theta\sigma\tau}{\gamma^\beta}\left(\frac{2\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+1-n)}{\beta}}\right)\Gamma\left(\frac{\beta+1-n}{\beta}\right) \tag{8}$$

$$E(x-\mu)^2 = \sum_{n=0}^{\infty}(-1)^n\binom{2}{n}\mu^n\left(\frac{\sigma\tau(1-\theta)}{\gamma^\beta}\left(\frac{\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+2-n)}{\beta}} + \frac{2\theta\sigma\tau}{\gamma^\beta}\left(\frac{2\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+2-n)}{\beta}}\right)\Gamma\left(\frac{\beta+2-n}{\beta}\right) \tag{9}$$

$$E(x-\mu)^3 = \sum_{n=0}^{\infty}(-1)^n\binom{3}{n}\mu^n\left(\frac{\sigma\tau(1-\theta)}{\gamma^\beta}\left(\frac{\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+3-n)}{\beta}} + \frac{2\theta\sigma\tau}{\gamma^\beta}\left(\frac{2\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+3-n)}{\beta}}\right)\Gamma\left(\frac{\beta+3-n}{\beta}\right) \tag{10}$$

$$E(x-\mu)^4 = \sum_{n=0}^{\infty}(-1)^n\binom{4}{n}\mu^n\left(\frac{\sigma\tau(1-\theta)}{\gamma^\beta}\left(\frac{\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+4-n)}{\beta}} + \frac{2\theta\sigma\tau}{\gamma^\beta}\left(\frac{2\tau\,\sigma}{\gamma^\beta}\right)^{-\frac{(\beta+4-n)}{\beta}}\right)\Gamma\left(\frac{\beta+4-n}{\beta}\right) \tag{11}$$

In a case where by $\mu = 0$, the expression $E(x-\mu)^r$ reduces to $E(X^r) = \mu_r'$, which denotes the rth moment about the origin as presented in Equation (5).

The variance (Second central moment) is obtained from the rth moment about the mean when $r = 2$.

When $r = 2$, the central moment becomes the variance and this is given as Equation (9) above

2.4.3 The coefficient of variation

The coefficient of variation (CV) measures relative variability with respect to the mean. It is a dimensionless metric useful for comparing datasets with different scales. It is defined as:

$$CV = \frac{\sqrt{\sigma^2}}{\mu}$$

Where $\mu \; and \; \sigma^2$ are given by equations (5) and (9)

2.4.4 Coefficient of Skewness

The coefficient of skewness (CS) measures the degree of asymmetry of a distribution about its mean. A positive value indicates right skewness, a negative value indicates left skewness, and zero indicates perfect symmetry. It is mathematically defined as:

$$CS = \frac{E\{(X-\mu)^3\}}{\{E[(X-\mu)^2]\}^{\frac{3}{2}}}$$

Where $E[(X-\mu)^2]$ and $E\{(X-\mu)^3\}$ are given by equations (9) and (10) respectively.

2.4.5 Coefficient of Kurtosis

The coefficient of kurtosis (CK) measures the peakedness and tail behaviour of a distribution relative to the normal distribution. High values indicate a leptokurtic (peaked, heavy-tailed) distribution, while low values indicate a platykurtic (flat) distribution. It is defined as:

$$CK = \frac{E[(X-\mu)^4]}{[E(X-\mu)^2]^2}$$

Where $[E(X-\mu)^2]$ and $E[(X-\mu)^4]$ are given in equations (9) and (11).

2.4.6 The Moment generating function of TE-NWP distribution

The moment generating function (MGF), denoted by $M_X(t)$, is a mathematical tool that encapsulates all information about a random variable and its moments. For the TE–NWP distribution, the MGF is given by:

$$M_X(t) = E(e^{tX}) \quad = e^{tE(X)}$$

By the Maclaurin series for e^t and substituting the expression for $E(X^r)$, we obtain:

$$M_X(t) = \sum_{r=0}^{\infty}\left(\frac{t^r\gamma^r\Gamma\left(1+\frac{r}{\beta}\right)}{(\tau\,\sigma)^{\frac{r}{\beta}}\,r!}\left(1-\theta+\frac{\theta}{2^{\frac{r}{\beta}}}\right)\right) \tag{12}$$

2.4.7 Reliability analysis

Reliability analysis assesses the probability that a system or component performs its intended function beyond a specified time.

2.4.7.1 Survival function of TE-NWP distribution

Let X denote the lifetime of a system or component with cumulative distribution function $F(x)$. The survival function (or reliability function) represents the probability that the system operates beyond time x.

$$S(x) + F(x) = 1$$

it implies that the survival function $(S(x))$ is given by

$$S(x) = 1 - F(x)$$

For the TE–NWP distribution, substituting the expression for $F(x)$ into the survival function and simplifying yields:

$$S(x) = exp - \left(\tau\sigma\left(\frac{x}{\gamma}\right)^{\beta}\right)\left(1 + \theta exp - \left(\tau\sigma\left(\frac{x}{\gamma}\right)^{\beta}\right)\right) \tag{13}$$

2.4.7.2 Hazard function of TE-NWP distribution

The hazard function $h(x)$, or failure rate, gives the instantaneous risk of failure at time x, given survival up to that time. It is defined mathematically as:

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)}$$

where $f(x)$ and $S(x)$ are given by equations (4) and (13).

Figures 3 and 4 display the different shapes of the survival and hazard functions of Transmuted

Exponential-New Weibull Pareto (TE-NWP) distribution for selected parameter values.

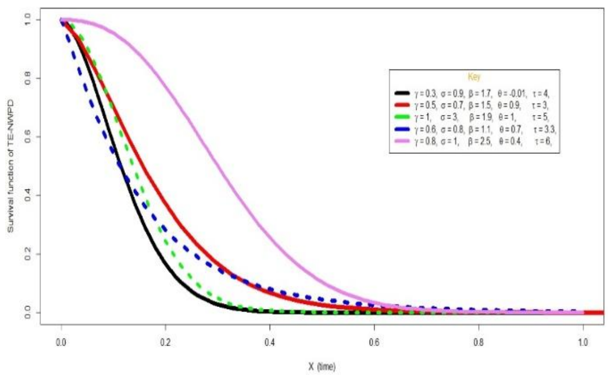

Figure 3: Plot of survival function for some selected values of TE-NWP distribution

Figure 3: The survival function of the Transmuted Exponential–New Weibull Pareto (TE–NWP) distribution decreases steadily from 1 toward 0 as time progresses, reflecting the gradual reduction in the probability of survival. The rate of decline varies with parameter values, demonstrating the distribution's ability to model systems with different reliability levels.

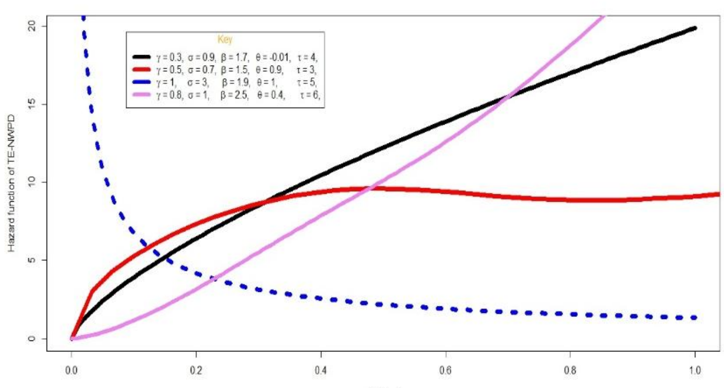

Figure 4: Plot of hazard function for some selected parameter values of TE-TE-NWP distribution.

Figure 4: The hazard function illustrates different structural forms, such as increasing, decreasing, and bathtub-shaped depending on the parameter settings. This flexibility highlights the robustness of the TE–NWP model in capturing various failure mechanisms, including early-life, random, and wear-out failures, confirming its suitability for modelling real-life reliability and lifetime data.

2.4.8 Quantile Function of TE-NWP Distribution

The quantile function is the inverse of the cumulative distribution function (CDF), giving the value of X corresponding to a cumulative probability p $(0 < p < 1)$. For the TE–NWP distribution, it is expressed as:

$$x_p = -\gamma\sigma^{\frac{-1}{\beta}} ln\left(\left[\frac{(\theta-1)\pm\sqrt{(\theta-1)^2+4\theta(1-q)}}{2\theta}\right]^{\frac{1}{\tau\beta}}\right) \quad (14)$$

$$for\ \theta \neq 0$$

2.4.9 The Median of TE-NWP Distributions

The median, $x_{0.5}$, which corresponds to the 50th percentile, is given by

$$x_{0.5} = -(\gamma\sigma)^{-\frac{1}{\beta}} \ln\left\{\left[\frac{(\theta-1)\pm\sqrt{(\theta-1)^2+2\theta}}{2\theta}\right]^{\frac{1}{\tau\beta}}\right\} \quad (15)$$

$$\theta \neq 0.$$

This expression provides the central value of the TE–NWP distribution.

3. Estimation of parameters of TE-NWP distribution

The estimation of the parameter of the proposed distribution is carried out using the method of maximum likelihood. Consider a random sample $x_1, x_2\, x_3, \ldots, x_n$ drawn from the TE-NWP distribution with an unknown parameter vector $\psi = (\beta, \gamma, \sigma, \sigma, \theta)^T$. To obtain the MLE of ψ, the corresponding log-likelihood function is given as:

$$\log Lf(x;,\psi) = nlog\tau + \sum_{i=1}^{n} \log h(x_i;\psi) + (\tau-1)\sum_{i=1}^{n} \log\left(1 - H(x_i;\psi)\right) + \sum_{i=1}^{n} \log\left(1-\theta+2\theta\left(1-\ H(x;\ \psi)\right)^{\tau}\right)$$

$$U_\tau = \frac{d\log Lf(x;\psi)}{d\tau} = \frac{n}{\tau} + \sum_{i=1}^{n} \log\left(1 - H(x_i;\psi)\right) + 2\theta \sum_{i=1}^{n} \frac{\left(1 - H(x;\ \psi)\right)^{\tau} ln\left(1 - H(x_i;\psi)\right)}{\left(1 - \theta + 2\theta\left(1 - H(x;\ \psi)\right)^{\tau}\right)}$$

$$U_\theta = \frac{d\log Lf(x;\psi)}{d\theta} = \sum_{i=1}^{n} \frac{2\left(1 - H(x;\ \psi)\right)^{\tau} - 1}{\left(1 - \theta + 2\theta\left(1 - H(x;\ \psi)\right)^{\tau}\right)}$$

$$U_\psi = \sum_{i=1}^{n} \frac{h'(x_i;\psi)}{h(x_i;\psi)} - (\tau - 1)\sum_{i=1}^{n} \frac{H'(x_i;\psi)}{\left(1 - H(x_i;\psi)\right)} - 2\theta\tau \sum_{i=1}^{n} \frac{\left(1 - H(x;\ \psi)\right)^{\tau-1} H'(x_i;\psi)}{\left(1 - \theta + 2\theta\left(1 - H(x;\ \psi)\right)^{\tau}\right)}$$

Where $H'(.)$ is the derivative of $H(x_i;\psi)$ with respect to ψ.

By setting each of the above nonlinear equations to zero and solving resulting system of equations simultaneously gives the MLE of $(\hat{\tau},\ \hat{\theta}\ ,\hat{\psi})^T$. Because the system of equations does not have closed forms, nonlinear optimization methods such as the Newton-Rapson algorithm was used to obtain their estimates and the results are shown in Table 1.

3.1 Simulation Study

In this section, a simulation study was carried out to evaluate the efficiency of the MLEs of the TE-NWP distribution parameters and the results are shown in Table 1.

3.2 Discussion of the simulation results

Table 1 presents the mean estimates, biases, variances, mean squared errors (MSEs) and standard error (SE) of the parameters of the Transmuted Exponential–New Weibull Pareto Distribution obtained using the Maximum Likelihood Estimation (MLE) method. The estimation was implemented through the Newton–Raphson algorithm using the optim() function in R for different sample sizes ($n = 20,100,200,500$). The results reveal that the parameter estimates are stable and move closer to their true values as the sample size increases, while both bias, MSE and SE values consistently decrease. This behavior confirms that the MLEs of the TE–NWP distribution parameters are consistent, asymptotically unbiased, and efficient**.** The findings demonstrate that the proposed estimation technique provides accurate and reliable parameter estimates even for small samples, and its performance improves with larger data sizes. This supports the suitability of the MLE method and the robustness of the TE-NWP model for analyzing lifetime data characterized by skewness and heavy tails.

4. Applications

Two real-life datasets were used to assess the goodness-of-fit of TE-NWP distribution

4.1. Real datasets

Dataset I: The data are the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place. These data were analyzed by Choulakian & Stephens (2011), Merovci, & Puka, (2014) to determining the distribution that has a better fit.

Dataset II: The data represents the life of a Kevlar 373/epoxy fatigue fracture subjected to constant pressure at 90% stress until it collapsed. The dataset was used by Abdul-Moniem and Seham (2015) in determining the flexibility of their proposed model.

4.2 Analysis of Dataset I

Table 2 shows that the skewness value of 1.473 confirms the right-skewed nature of Dataset I, indicating that a few large observations pull the mean toward the right. Additionally, the kurtosis value of 5.889 suggests a leptokurtic distribution, characterized by a pronounced peak and heavier-than-normal tails, implying the presence of extreme values and possible outliers.

Table 3 demonstrates that the proposed TE-NWP distribution provides a superior fit compared with the competing distributions. This is evidenced by its lower log-likelihood (−LL), Akaike Information Criterion (AIC), Hannan–Quinn Information Criterion (HQIC), Consistent Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) values. In addition, the goodness-of-fit tests: Cramér–von Mises (CM), Anderson–Darling (AD), and Kolmogorov–Smirnov (K–S) further confirm the adequacy and superiority of the proposed model.

4.3 Analysis of Dataset II

Table 4 indicates that Dataset II is right-skewed, as reflected by the skewness value of 1.980, showing that a few large observations pull the mean toward higher values. The kurtosis value of 8.161 points to a leptokurtic distribution, with a pronounced peak and heavy tails, suggesting the presence of extreme values and possible outliers.

Table 5 shows parameter estimates, information criteria, and goodness-of-fit statistics for several distributions fitted to Dataset II. The TE–NWP model achieves the lowest values for log-likelihood, AIC, HQIC, CAIC, and BIC, as well as smaller Cramér–von Mises, Anderson–Darling, and Kolmogorov–Smirnov statistics, indicating the best overall fit. Confirming TE–NWP as the most suitable model for this dataset.

4.4 Discussion and Results

The study proposed the Transmuted Exponential–New Weibull Pareto (TE–NWP) distribution**,** a flexible five-parameter lifetime model suitable for skewed and heavy-tailed data. Its PDF and CDF are mathematically valid and capable of modelling diverse hazard rate behaviours, including increasing, decreasing, and bathtub-shaped patterns. The distribution's statistical properties, such as moments, skewness, kurtosis, survival, hazard, and quantile functions, were derived to describe its behaviour comprehensively.

Graphical illustrations (Figures 1–4) demonstrates the model's flexibility in capturing right-skewed lifetime data and realistic reliability scenarios. Parameters estimated via Maximum Likelihood Estimation showed stability and consistency, with simulation results (Table 1) confirming decreasing bias and mean squared errors with increasing sample size.

Applications to real-life datasets-flood peak exceedances in Canada and Kevlar fatigue lifetimes—revealed that the TE–NWP distribution provides superior fits compared to alternative models, effectively capturing the skewed and heavy-tailed characteristics of the data (Tables 3 and 5). Overall, the TE–NWP distribution is robust, versatile, and well-suited for practical lifetime and reliability data modelling.

5. CONCLUSION

This study proposed a flexible five-parameter model, the Transmuted Exponential–New Weibull Pareto (TE–NWP) distribution, tailored for positively skewed and heavy-tailed lifetime data. Key statistical properties, including moments, reliability measures, and the quantile function, were derived, and the model parameters were efficiently estimated using the maximum likelihood method. Simulation results demonstrated that the estimators are consistent, with bias and mean squared error decreasing as sample size increases. Applications to two real-life datasets confirmed that the TE–NWP distribution provides a superior fit relative to several competing models, as evidenced by information criteria and goodness-of-fit assessments. Overall, the TE–NWP distribution shows strong potential for reliability analysis, risk modelling, and survival studies, offering enhanced flexibility and practical applicability. Future research could explore its application to diverse datasets in fields such as finance, insurance, climatology, hydrology, and biomedical sciences, where skewness and heavy-tailed behaviour are common.

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Table 1: Mean of the MLEs, Biases, Variances, Mean square errors (MSEs) and Standard errors (SE) of the TE-NWP distribution for $\gamma = 0.3,\ \sigma = 0.9,\ \beta = 1.7,\ \theta = 0.01, \tau = 4$

Sample size (n)	Parameter	Estimate	Bias	Variance	MSE	SE
n = 20	$\hat{\gamma}$	0.5424990	0.2425	0.04551728	0.10432353	0.2134
	$\hat{\sigma}$	0.8596329	−0.0404	0.07333169	0.074961192	0.2708
	$\hat{\beta}$	0.9227461	−0.7773	0.7913913	1.395514925	0.8896
	$\hat{\theta}$	-0.2375976	−0.2476	0.13600442	0.187805087	0.3688
	$\hat{\tau}$	3.9902565	−0.0097	0.71049335	0.710588285	0.8435
n = 100	$\hat{\gamma}$	0.47052935	0.1705	0.022883993	0.051964252	0.1513
	$\hat{\sigma}$	0.86176369	−0.0382	0.03297489	0.034436905	0.1816
	$\hat{\beta}$	0.97735741	−0.7226	0.5182942	1.040506513	0.7209
	$\hat{\theta}$	-0.2134580	−0.2235	0.05679256	0.098187717	0.2383
	$\hat{\tau}$	3.99128263	−0.0087	0.58081613	0.580853219	0.7610
n = 200	$\hat{\gamma}$	0.4691840	**0.1692**	0.02154363	0.050166855	0.1468
	$\hat{\sigma}$	0.8743535	−0.0256	0.023458560	0.024116302	0.1531
	$\hat{\beta}$	1.1276209	−0.5724	0.13093642	0.458554254	0.3619
	$\hat{\theta}$	-0.02492024	−0.0349	0.029808923	0.030031536	0.1726
	$\hat{\tau}$	3.9939099	−0.0061	0.35730459	0.357330499	0.5977
n = 500	$\hat{\gamma}$	0.44052935	0.1405	0.01032812	0.030076618	0.1016
	$\hat{\sigma}$	0.89176369	−0.0082	0.01333169	0.013399526	0.1155
	$\hat{\beta}$	1.37735741	−0.3226	0.04182942	0.14590018	0.2045
	$\hat{\theta}$	-0.01492024	−0.0249	0.01808923	0.018113438	0.1345
	$\hat{\tau}$	3.99524261	−0.0048	0.15730459	0.157327222	0.3966

Dataset 1

1.7, 2 .2, 1 4.4, 1.1, 0.4, 20.6, 5.3, 0.7, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6,14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0, 1.9, 2.8.

Dataset II

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 2: Descriptive statistics for dataset I

N	Min	Max	Mean	Variance	1st Quantile	Median	3rd Quantile	Skewness	Kurtosis
72	0.100	64.000	12.204	151.222	2.125	9.500	20.125	1.473	5.889

Table 3: Estimates of the parameters and model selection criteria for dataset 1

Distribution	Parameters Estimates	Statistical Test							
		-LL	AIC	HQIC	CAIC	BIC	CM	AD	K-S
TE-NWP	$\hat{\gamma} = 0.81171$ $\hat{\sigma} = 0.307260$ $\hat{\beta} = 0.894271$ $\hat{\theta} = -0.037519$ $\hat{\tau} = 0.30726$	251.49	512.99	517.52	513.74	524.37	0.1476	0.8397	0.1057
TE-E	$\hat{\alpha} = 1.195447$ $\hat{\beta} = 0.089883$ $\hat{\lambda} = -0.030018$	254.69	515.75	518.12	515.66	524.39	0.3952	2.9326	0.1814
NWP	$\hat{\beta} = 1.299869$ $\hat{\sigma} = 94.986374$ $\hat{\gamma} = 401.63809$	261.00	528.97	531.69	529.24	535.80	0.6515	5.6076	0.2085
WD	$\hat{\beta} = 0.507927$ $\hat{\eta} = 79.578373$	296.64	597.28	599.09	597.39	601.83	5.4711	24.948	0.4845
EP	$\hat{\beta} = 699.13673$ $\hat{\theta} = 3.281094$ $\hat{\alpha} = 121.49304$	309.79	625.59	628.32	625.86	632.43	3.8901	72.470	0.4121
TEG	$\hat{\lambda} = -0.50000$ $\hat{\theta} = 1.254828$ $\hat{\alpha} = 0.192654$	638.00	1282.0	1284.7	1282.3	1288.8	4.7476	188.16	0.5168
TGIW	$\hat{\alpha} = 1.474999$ $\hat{\beta} = 0.710723$ $\hat{\gamma} = 1.765301$ $\hat{\lambda} = -0.728873$	264.99	537.98	541.61	538.45	547.09	0.3604	2.2537	0.1499

Table 4: Descriptive statistics for dataset II

N	Min	Max	Mean	Variance	1st Quantile	Median	3rd Quantile	Skewness	Kurtosis
76	0.0251	9.0960	1.9592	2.4774	0.9048	1.7362	2.2956	1.9796	8.1609

Table 5: Estimates of the parameters and model selection criteria for dataset 1I

Distribution	Parameters Estimates	Statistical test							
		-LL	AIC	HQIC	CAIC	BIC	CM	AD	K-S
TE-NWP	$\hat{\gamma} = 1.752291$ $\hat{\sigma} = 0.233252$ $\hat{\beta} = 1.035729$ $\hat{\theta} = -0.036670$ $\hat{\tau} = 0.760210$	414.142	838.284	844.078	838.544	852.54	0.757	4.094	0.138
TE-E	$\hat{\alpha} = 2.476575$ $\hat{\beta} = 0.160051$ $\hat{\lambda} = -0.387831$	428.345	862.689	866.166	862.833	871.25	1.539	7.159	0.185
NWP	$\hat{\beta} = 1.235585$ $\hat{\sigma} = 99.946115$ $\hat{\gamma} = 400.01637$	418.241	842.483	845.956	842.627	851.04	1.622	7.732	0.187
WD	$\hat{\beta} = 0.504703$ $\hat{\eta} = 79.739988$	520.370	1044.74	1047.58	1044.80	1050.44	12.480	56.35	0.518
EP	$\hat{\beta} = 699.264156$ $\hat{\theta} = 2.930493$ $\hat{\alpha} = 116.549355$	502.693	1011.38	1014.86	1011.53	1019.94	15.395	151.9	0.551
TE-G	$\hat{\lambda} = -0.50000$ $\hat{\theta} = 2.397137$ $\hat{\alpha} = 0.285286$	896.463	1798.93	1802.41	1799.08	1807.49	7.653	133.3	0.403
TGIW	$\hat{\alpha} = 1.30621$ $\hat{\beta} = 0.835703$ $\hat{\gamma} = 1.954000$ $\hat{\lambda} = -0.855676$	436.678	881.356	885.991	881.614	892.764	1.249	7.494	0.155