EWMA-TYPE CHART BASED ON REGRESSION ESTIMATOR USING AUXILIARY VARIABLE AND THE GENERALIZED LIKELIHOOD RATIO TEST STATISTIC

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Abstract

Statistical process control (SPC) is an important integral aspect of every manufacturing process whose aim is to maintain product or service standard. Control charts are the most indispensable aspect of the SPC because of their statistical background. The cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA) charts are better options to the Shewhart $\overline{X}$-charts owing to their abilities to timely detect small and moderate shifts in the process parameter. To increase the sensitivity of the EWMA chart, the use of auxiliary variables in the estimation of its charting statistic has been proposed. Also, to make the chart monitor for both the location and dispersion parameters of the process concurrently, its charting statistic has been modified by using the generalized likelihood ratio statistic. This work proposes a generalized likelihood ratio statistic which would use auxiliary variable that is based on regression estimation technique for joint monitoring of the process parameters. The average run length (ARL) and the median run length (MRL) of the chart were evaluated in simulation and the ARLs compared with those of the rival control charts in the literature. It was noticed that the proposed chart outperformed rival charts in detecting assignable causes of variations in the process.

Keywords: auxiliary information; average run length; control charts; generalized ratio statistic.

I. Introduction

Control chart is the most important and commonly used tool in the statistical process control (SPC) toolkit due to its statistical framework [1]. The major objective of the control chart is to quickly signal the assignable causes present in the system so that proper investigations are made and corrective decisions taken before any non-conforming items are produced [2]. Therefore, in SPC, to improve the quality of a process output demands designing a quality control chart that guarantees timely detection of unnatural variations irrespective of the magnitude. Unfortunately, the Shewhart- type control charts, though very popular because of their simplicity, have a major setback of being inefficient in detecting small shifts in the process parameters. This is simply because, in the controlling mechanism of the Shewhart-type control charts, the information contained in the lagged observation(s) is ignored, hence, they are described as memory-less-type control charts in the literature, see [3-8]. To make up for this shortcoming, Page [3] proposed the CUSUM chart and [4] proposed EWMA charts. These two control charts, unlike the Shewhart-type control charts, make use of the information contained in the lagged observation(s) as well as the current observation(s) and are described as memory-type control charts in the literature. On this premise, control charts are classified into two categories; the memoryless (Shewhart)-type and the memory-type control charts with respect to their design structures [9].

However, for years now, the EWMA control chart is considered as a better alternative to the Shewhart control chart in detecting small to moderate disturbances in the process parameters such as the mean or variance [10,11]. Over the years, a lot of works have been done in the literature which are targeted to further enhance the performance of the EWMA control chart by using auxiliary variables and such charts are described in the literature as auxiliary information based EWMA (AIB-EWMA) charts. AIB charts are mainly based on improving the accurateness of the estimators with auxiliary information (auxiliary variables) alongside the quality characteristic (study variable) in an attempt to enhance the detection ability of the charts when compared with those that are not using auxiliary information, see [12-14]. On this note, [9] proposed an EWMA-type chart denoted by AEWMA-chart which is based on a regression estimator with an auxiliary variable for monitoring the process location. The superiority of their proposed chart over the classical EWMA-chart was established in the work in terms of the average run length (ARL), where ARL is defined as the expected number of samples until a shift is signaled. ARL_0 and ARL_1 are the in-control (IC) and out-of-control (OOC) ARLs respectively. [15] proposed an EWMA-type chart, named TAEWMA chart, based on a regression estimator with two auxiliary variables for monitoring the process location. The superiority of their proposed chart over the classical and AEWMA charts was established in the work in terms of the ARL. The idea of auxiliary information is common in the field of sample survey and estimation techniques, see [16,17] and the regression estimation based auxiliary variable is one of the most efficient among its counterparts, see [9,15,18-21].

Many SPC procedures, including the works of [9], are designed to detect a shift from the location parameter (the mean) of the process. However, it is important to monitor for the dispersion parameter simultaneously considering that the control limits of the location charts depend on the variation of the process [22]. Therefore, it is desirable to design control charts that can be sensitive to both the location and dispersion parameters. Two major methods exist in the literature for doing this. It is either that two control charts that are designed for respective monitoring of the mean and the variance are combined for joint monitoring of the two parameters or that an omnibus-type test is used to design a single control scheme for detecting shifts in both the mean and the variance of the process, see [23-25] and [26-28] for the first and the second methods respectively. In line with the second method, [22] proposed a single chart called ELR_t-EWMA that integrates the EWMA procedure with the generalized likelihood ratio (GLR) test statistics for jointly monitoring of both the process mean and variance of the process. The performance of the ELR_t-EWMA chart was compared with the competing control charts (MEW, WLC and NCS) and it was discovered that when the mean and the variance of the process shift simultaneously, the ELR_t-EWMA chart generally performances better than the MEW-chart. The comparison with the

WLC and NCS charts showed that the ELR_t-EWMA chart offered more balanced protection to the various combinations of shifts. [15] proposed a chart called M_dELR_t-EWMA chart to improve on the work of [22] by including one auxiliary variable in its design using the estimator in [29]. The performance of the chart was compared with the ELR_t-EWMA chart at different values of weighting parameter (λ), correlation coefficient (ρ) and ARL_0. In all cases, M_dELR_t-EWMA chart outperformed the ELR_t-EWMA chart. In this research, the works of [22] and [15] will be improved on by including one auxiliary variable in the design but by using the regression-based estimator of [16] and the chart will be named M_RELR_t-EWMA chart.

II. Materials and Methods

In developing the proposed M_RELR_t-EWMA control chart, a Cochran-type regression estimator [16] is used in place of the two already considered estimators and the results combined through the generalized likelihood ratio statistics into a single statistic. The evaluation of the performance of the proposed chart and its application were implemented through simulation using Rsoftware.

The design structure of the ELR_t-EWMA control chart

The interest in any production process is on the quality characteristic. Suppose that $Y_{tj}; t = 1,2,3,..$ and $j = 1,2,3,\dots,n$ is a quality characteristic of interest taken from a normal distribution with mean μ_0 and variance σ_y^2. Roberts [4] proposed the EWMA statistic to find the mean of the t^{th} sample of the quality characteristic of the production process by

$$T_t = \lambda\overline{Y_t} + (1-\lambda)T_{t-1} \quad (1)$$

where the sample mean of the t^{th} sample is $\overline{Y_t}$ with sample size n and λ is the smoothing constant or the weight assignment parameter chosen such that $0 < \lambda \leq 1$. For an IC production process, [4] gave the mean and the variance of T_t as; $E(T_t) = \mu_0$ and $Var(T_t) = \sigma_{T_t}^2 = \sigma_{\bar{y}}^2\left[\frac{\lambda}{2-\lambda}\{1-(1-\lambda)^{2t}\}\right]$ respectively, where $\sigma_{\bar{y}}$ is the standard deviation of $\overline{Y}$. Therefore, the control limits are; Upper control limit (UCL) = $\mu_0 + L\sigma_{\bar{y}}\sqrt{\frac{\lambda}{2-\lambda}\{1-(1-\lambda)^{2t}\}}$, Center limit (CL)= μ_0 and the Lower control limit (LCL) = $\mu_0 - L\sigma_{\bar{y}}\sqrt{\frac{\lambda}{2-\lambda}\{1-(1-\lambda)^{2t}\}}$.

The above-described chart monitors for only the process mean. To arrive at the EWMA-type chart that concurrently monitors for shifts in both the mean and variance of the process, Zhang et al. [22] proposed a likelihood ratio based EWMA chart. The construction of the chart is done by using two EWMA statistics that are meant to detect shifts in the mean and the variance and they are given in (2) and (3) respectively.

$$U_t = \lambda\overline{y_t} + (1-\lambda)U_{t-1} \quad (2)$$

$$V_t = \lambda S_t^2 + (1-\lambda)V_{t-1} \quad (3)$$

where $\overline{y_t}$ is the sample mean at the time t, $S_t^2 = \frac{\sum_{j=1}^{n}(y_{tj}-U_t)^2}{n}$, $U_0 = 0, V_0 = 1$ and λ is as already defined. By using the generalized likelihood ratio statistics, [22] combined (2) and (3) into a single statistic which is capable of monitoring both the mean and the variance of a process simultaneously. The statistic is given in (4).

$$ELR_t = U_t + V_t - ln(V_t), t = 1,2,\ldots \quad (4)$$

The chart raises OOC alarm if $ELR_t > h$, where h is chosen to achieve a specified IC ARL.

The design Structure of the M_dELR_t-EWMA control chart

Let (Y,X) follow a bivariate normal distribution, that is, $(Y,X) \sim N_2(\mu_Y, \mu_X, \sigma_Y^2, \sigma_X^2)$, where Y is the quality characteristic (study variable) and X is an auxiliary variable which is assumed to be known. The principle of the design and construction of the ELR_t-EWMA chart is applicable in M_dELR_t-EWMA chart only that one auxiliary variable (X) is now used to estimate $\overline{y_t}$ in (2) by using the estimator in [29]. The new estimator is given by $M_d = \frac{\overline{y}+K(\mu_X-\overline{x})}{(a\overline{x}+b)}(a\mu_X + b)$, where a and b are the population parameters of the auxiliary variable X, $\overline{y}$ and $\overline{x}$ are the sample means of the study and the auxiliary variables, μ_X is the population mean of the auxiliary variable, $K = b_{yx} - \frac{a\overline{y}}{a\mu_X+b}$ and b_{yx} is the regression coefficient. By using the M_d estimator in place of $\overline{y_t}$, [15] redefined the statistics in (2) and (3) to be as in (5) and (6).

$$U_{dt} = \lambda M_d + (1-\lambda)U_{dt-1} \quad (5)$$

$$V_{dt} = \lambda S_{dt}^2 + (1-\lambda)V_{dt-1} \quad (6)$$

where M_d and λ are as already defined, $S_{dt}^2 = \frac{\sum_{j=1}^{n}(y_{tj}-U_{dt})^2}{n}$, $U_0 = 0, V_0 = 1$. By using the generalized likelihood ratio statistics (5) and (6) were combined into a single statistic capable of monitoring for both the mean and variance of a process simultaneously. The statistic is given in (7)

$$M_dELR_t = U_{dt} + V_{dt} - ln(V_{dt}), t = 1,2,\ldots \quad (7)$$

The chart raises OOC alarm if $M_dELR_t > h$, where h is chosen to achieve a specified IC ARL.

Design Structure of the proposed M_RELR_t-EWMA chart

Let (Y,X) follow a bivariate normal distribution, that is, $(Y,X) \sim N_2(\mu_Y, \mu_X, \sigma_Y^2, \sigma_X^2)$, where Y is the quality characteristic (study variable) and X is an auxiliary variable which is assumed to be known. The principle of the design and construction of the ELR_t-EMMA and the M_dELR_t-EWMA charts will be applied in the proposed M_RELR_t-EWMA chart but the estimation of $\overline{y_t}$ in (2) with an auxiliary variable (X) is now done by using the estimator in [16] which is given by $M_R = \overline{y} + b_{yx}(\mu_X - \overline{x})$, where $\overline{y}$ and $\overline{x}$ are the sample means of the quality characteristic and the auxiliary variable respectively, μ_X is the population mean of the auxiliary variable and b_{yx} is the regression coefficient. By using the M_R estimator in place of $\overline{y_t}$ and M_d, we redefine the statistics in (2) and (3) to be as in (8) and (9).

$$U_{Rt} = \lambda M_R + (1-\lambda)U_{Rt-1} \quad (8)$$

$$V_{Rt} = \lambda S_{Rt}^2 + (1-\lambda)V_{Rt-1} \quad (9)$$

where $S_{Rt}^2 = \frac{\sum_{j=1}^{n}(y_{tj}-U_{Rt})^2}{n}$, $U_0 = 0, V_0 = 1$. By using the generalized likelihood ratio statistics, we combine (8) and (9) into a single statistic that will be capable of monitoring for both the mean and variance of a process simultaneously. The statistic is as given in (10)

$$M_RELR_t = U_{Rt} + V_{Rt} - ln(V_{Rt}), t = 1,2, \ldots \quad (10)$$

The chart raises OOC alarm if $M_RELR_t > h$, where $h$is chosen to achieve a specified IC ARL.

III. Results

Simulation study of the proposed chart

In this section, we discuss the performance of the M_RELR_t-EWMA chart in terms of the ARL and the MRL of the run length (RL)distribution. It is a good practice to present the MRL performance of a control chart alongside the ARL considering the fact that MRL measures the center of a distribution better when the distribution is skewed, which is the case with RL distribution. A chart with the smallest ARL_1 when compared to other similar charts performing under the same condition is said to have a better performance. The simulation of the ARL and MRL of the chart is obtained through the following guidelines:

a. Let Y denote the quality characteristic of a production process which follows a normal distribution with mean, $\mu = \mu_0 + \alpha\sigma_0$ and variance, $\sigma^2 = \beta^2\sigma_0^2$, where μ_0 and σ_0^2 are the known process mean and variance, the process is IC if and only if $\alpha = 0$ and $\beta = 1$ and OOC if otherwise. The values of h(the UCL) of the chart for the different levels of correlations between the quality characteristic (Y) and the auxiliary variable (X) are obtained and tabulated in Table 2 in line with the specified IC ARL_0 and at a particular value of the parameter (λ).

b. Simulate a random sample of size $n = 5$ from a bivariate normal distribution, such that $(Y, X) \sim N_2(0,10,1,2,\rho)$.

c. By using the observations simulated, calculate the value of the estimator M_R.

d. Select the values of λ and h according to the desired in-control ARL_0 from Table 2.

e. Plot the values of the charting statistic M_RELR_t on the control chart, that is, compare the values with the corresponding value of h and count the number of points until an OOC signal is triggered by the chart.

f. Repeat the above steps 50,000 times and calculate the ARL and MRL of the process. The results for $\lambda = 0.2$ are presented in Tables 3 and 4 for ARL_0 of 185 and 370 respectively.

The proposed M_RELR_t chart is compared with the ELR_t and M_dELR_t charts in terms of the ARL when the sample size $(n = 5)$ and $\lambda = 0.1$ and 0.2. The conventional ARL_0 of 185 and 370 were considered. Table 2 contains the upper control limits of the M_RELR_t chart for the various ARL_0 and the correlation coefficients (ρ) at $\lambda = 0.2$. Table 3 contains the ARLvalues of the M_RELR_t chart for $n = 5$, $\lambda = 0.2$ and $ARL_0 = 185$ for the different values of the correlation coefficient (ρ), Table 4 contains the values for $n = 5$, $\lambda = 0.2$ and $ARL_0 = 370$ for the different values of the correlation coefficient (ρ), Table 5 contains the values for $n = 5$, $\lambda = 0.1$ and $ARL_0 = 370$ for different values of the correlation coefficient (ρ).

Numerical Application of the proposed M_RELR_t chart

The applicability of the M_RELR_t chart is demonstrated with simulated data from a bivariate normal distribution. The simulation and subsequent application are done as follows;

1. Simulate twenty (20) samples each of size five (5) from $(Y, X) \sim N_2(\mu_Y = \mu_0 + \alpha\sigma_0, \mu_X = 10, \sigma_Y^2 = \beta^2\sigma_0^2, \sigma_X^2 = 2, \rho = 0.5)$, where the first ten samples are when $\alpha = 0$and $\beta = 1$, that is, IC samples. The second set of the samples are when $\alpha = 0.25$and $\beta = 0.25$; that is, the OOC samples. To ensure reproducible samples, a seed was set at set.seed (1) in R statistical software to carry out the simulation.
2. Compute the charting statistics $(M_R ELR_t)$ of the $M_R ELR_t$ chart for the 20 samples and compare the values with $h = 1.2353$, which is the upper control limit of the chart when the correlation coefficient $\rho = 0.50$. When a point falls below $h = 1.2353$, the process is IC at that point and, as such, no signal is raised by the chart. However, the chart raises an OOC alarm once a point falls above $h = 1.2353$.

The sample numbers and the values of the charting statistics for the different samples are presented in Table 1 and each compared with $h = 1.2353$.

Table 1: Samples with the corresponding values of the charting statistics of the $M_R ELR_t$-chart for $\lambda = 0.2$ and $ARL_0 = 185$

Sample Number	$M_R ELR_t$	< or > $(h = 1.2353)$.
1	1.0079	IC
2	1.0702	IC
3	1.0022	IC
4	1.0053	IC
5	1.0257	IC
6	1.0217	IC
7	1.0174	IC
8	1.0221	IC
9	1.0358	IC
10	1.0096	IC
11	1.0215	IC
12	1.0766	IC
13	1.1596	IC
14	1.2622	OOC
15	1.3889	OOC
16	1.5097	OOC
17	1.6015	OOC
18	1.7693	OOC
19	1.9182	OOC
20	1.9618	OOC

Table 2: The UCL of the $M_d ELR_t$-EWMA chart for different ARL_0 and levels of correlation coefficients(ρ) and lambda $(\lambda) = 0.2$

	Levels of correlation			
ARL_0	$\rho = 0$	$\rho = 0.25$	$\rho = 0.50$	$\rho = 0.75$
ARL_0=185	1.26551	1.25632	1.2353	1.1951
ARL_0=370	1.3235	1.3212	1.2881	1.2141
ARL_0=500	1.4500	1.4499	-0.5812	0.7312

Table 3: The ARL AND MRL of the $M_R ELR_t$-EWMA chart for λ = 0.2 and $ARL_0 = 185$

		Levels of Shift (α)													
		0.00		**0.25**		**0.5**		**0.75**		**1.00**		**1.30**		**3.00**	
β	ρ	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**
0.25	**0**	4.01	4	3.98	3.5	3.98	3.5	3.38	3	2.6	2	1	0	0	0
	0.25	3.98	3.5	3.85	3.5	3.86	3.5	3.33	2.5	2.467	2	1	1	0	0
	0.5	3.97	3.5	3.85	3.5	3.56	3	3.22	2.5	2.067	2	1	1	0	0
	0.75	3.68	3	3.50	3	3.34	3	2.667	2	2.0	2	1	1	0	0
0.5	**0**	0	5.49	5	5.24	5	4.56	4	3.4	3	2.667	2	1.2	1	0
	0.25	0.25	5.40	5	5.10	5	4.48	4	3.467	3	2.933	2	1.067	1	0
	0.5	0.5	5.16	4.5	4.89	5	4.27	4	2.933	3	2.40	2	1	1	0
	0.75	0.75	4.89	4.5	4.76	5	4.02	3.5	2.733	3	1.933	2	0.933	1	0
1.00	**0**	184.52	182	26.80	25	8.30	7	4.733	4	2.067	2	0.933	1	0	0
	0.25	184.46	182	25.34	24.5	7.62	7	4.733	4	2.267	2	1	1	0	0
	0.5	184.11	182	22.99	21	7.30	7	3.933	4	2.067	2	0.6	1	0	0
	0.75	184.44	182	22.80	21	6.91	6	3.467	3	1.667	2	0.533	1	0	0
1.25	**0**	16.01	16	10.90	10	6.15	6	4.333	3	1.933	1	0.80	1	0	0
	0.25	15.41	9	10.39	8	5.90	5	3.533	3	2.067	2	0.667	1	0	0
	0.5	14.23	10	9.52	7	5.39	5	3.333	3	1.667	2	0.533	1	0	0
	0.75	13.0	8	8.76	6	5.22	5	2.4	2	1.267	1	0.4	0	0	0
1.5	**0**	3.4	3	3.267	3	3.133	3	2.067	2	1.133	1	0.6	1	0	0
	0.25	4.467	3	3.6	3	2.8	2	1.8	1	1.267	1	0.6	1	0	0
	0.5	4	3	3.733	3	3	2	1.667	1	1.267	1	0.6	1	0	0
	0.75	4.2	3	3.4	3	2.2	2	1.467	1	0.933	1	0.467	1	0	0
2.00	**0**	0	1.733	1	1.267	1	0.867	1	0.733	1	0.60	1	0.467	0	0
	0.25	0.25	1.067	1	0.733	1	0.733	1	0.733	1	0.533	1	0.467	0	0
	0.5	0.5	1.067	1	0.733	1	0.733	1	0.733	1	0.533	1	0.400	0	0
	0.75	0.75	1.067	1	0.733	1	0.733	1	0.733	1	0.533	1	0.400	0	0

Table 4: The ARL AND MRL of the $M_R ELR_t$-EWMA chart for λ= 0.2 and $ARL_0 = 370$

		Levels of Shift (α)													
		0.00		**0.25**		**0.5**		**0.75**		**1.00**		**1.30**		**3.00**	
β	ρ	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**	**ARL**	**MRL**
0.25	**0**	4.15	4.00	4.06	4.00	4.02	4.00	3.97	3.00	3.02	2.50	2.00	2.00	0	0
	0.25	4.05	4.00	4.03	3.90	4.00	4.00	3.74	3.00	3.07	2.50	2.01	2.00	0	0
	0.5	4.00	3.50	4.02	4.00	3.85	3.50	3.48	3.00	3.05	2.50	2.00	2.00	0	0
	0.75	3.97	3.50	3.86	3.50	3.78	3.00	3.32	3.00	2.79	2.00	1.99	1.50	0	0
0.5	**0**	6.05	6.00	5.74	5.50	5.01	5.00	4.11	4.00	3.32	3.00	2.13	2.00	0	0
	0.25	5.96	6.00	5.65	5.50	4.95	4.50	4.02	3.50	3.25	3.00	2.12	2.00	0	0
	0.5	5.72	5.50	5.29	5.00	4.66	4.50	3.78	3.50	3.10	3.00	2.01	1.50	0	0
	0.75	5.42	5.20	5.09	5.00	4.42	4.50	3.50	3.00	2.91	2.50	1.89	1.50	0	0
1.00	**0**	370.50	369	37.30	35	9.66	9	5.11	5	3.40	3.0	1.84	1	0	0
	0.25	370.31	368	34.22	33	9.32	9	4.80	4.5	3.35	3.0	1.69	1	0	0
	0.5	370.20	369	31.99	32	8.40	8	4.50	4.5	3.10	3.0	1.50	1	0	0
	0.75	370.03	369	32.07	32	7.53	7.5	4.08	4	2.50	2.0	1.32	1	0	0
1.25	**0**	19.60	18	13.00	13	6.89	6	4.21	4	2.83	2	1.87	1	0	0
	0.25	19.00	18	12.46	12	6.65	6	4.10	4	2.50	2	1.62	1	0	0
	0.5	18.01	17	10.85	9	6.13	6	3.40	3	2.21	2	1.49	1	0	0
	0.75	15.64	15	10.33	9	5.54	5	2.99	3	2.07	2	1.36	1	0	0
1.5	**0**	5.97	5	5.36	5	4.32	4	3.21	3	2.21	2	1.50	1	0	0
	0.25	5.78	5	5.36	5	4.24	4	3.05	3	2.10	2	1.32	1	0	0
	0.5	5.55	4	4.99	4	3.92	3	2.94	2	1.97	1	1.30	1	0	0
	0.75	5.17	4	4.65	4	3.64	3	2.40	2	1.44	1	1.04	1	0	0
2.00	**0**	2.38	2	2.32	2	2.10	2	1.87	1	1.32	1	1.03	1	0	0
	0.25	2.21	2	2.02	2	1.98	1	1.63	1	1.28	1	1.01	1	0	0
	0.5	2.01	2	1.87	1	1.55	1	1.43	1	1.22	1	1.00	1	0	0
	0.75	1.96	1	1.50	1	1.32	1	1.20	1	1.01	1	1.00	1	0	0

Table 5: The ARL AND MRL of the $M_R ELR_t$-EWMA chart for λ= 0.1 and $ARL_0 = 370$

		ELR_t	$M_R ELR_t$ ARL(MRL)			
			$\rho = 0.00$	$\rho = 0.25$	$\rho = 0.50$	$\rho = 0.75$
α	β	UCL=1.1052	UCL=1.1315	UCL=1.1220	UCL=1.1140	UCL=1.0912
0	0.25	5.80	5(5)	5(5)	5(5)	4(4)
	0.75	16.39	16.47(16)	16.29(16)	15.40(14)	14.20(12)
	1.00	379.65	371.58(329.00)	370.85(373.5)	367.94(364.5)	370.41(275.5)
	1.25	16.92	16.88(12.50)	14.06(14.50)	11.88(14.00)	13.68(9.00)
	1.50	6.17	5.38(4.50)	4.88(4.00)	4.88(4.00)	4.44(3.00)
0.50	0.25	5.05	5.03(5.00)	5.05(5.00)	4.81(5.00)	4.00(4.00)
	0.75	8.30	8.25(8.00)	8.01(7.00)	7.52(7.00)	7.01(7.00)
	1.00	9.77	9.68(9.50)	9.43(9.00)	8.70(8.00)	7.70(7.00)
	1.25	7.08	7.06(6.00)	7.00(7.00)	6.48(6.00)	5.06(5.00)
	1.50	4.61	3.88(3.50)	3.25(3.00)	3.43(3.00)	3.13(3.00)
1.00	0.25	4.00	3.56(3.00)	3.56(3.00)	3.19(3.00)	3.00(3.00)
	0.75	4.21	3.50(4.00)	3.43(4.00)	3.31(3.50)	2.75(3.00)
	1.00	3.98	3.38(4.50)	3.06(3.00)	3.00(3.00)	2.31(2.00)
	1.25	3.51	2.75(2.50)	2.69(2.00)	2.50(2.00)	1.88(2.00)
	1.50	2.93	1.93(1.00)	1.81(1.00)	1.69(1.00)	1.56(1.00)
1.50	0.25	2.88	2.00(2.00)	2.00(2.00)	1.81(2.00)	1.06(1.00)
	0.75	2.55	1.50(1.50)	1.50(1.50)	1.43(1.00)	1.13(1.00)
	1.00	2.41	1.31(1.00)	1.28(1.00)	1.38(1.00)	0.81(1.00)
	1.25	2.25	1.19(1.00)	1.13(1.00)	0.88(1.00)	0.69(1.00)
	1.50	2.05	1.00(1.00)	0.75(1.00)	0.81(1.00)	0.68(1.00)

Discussion of result

It can be observed from Tables 3-5 that as the correlation coefficient increases, the proposed chart raises more OOC alarms by returning smaller OOC $ARL$and MRL. For instance, consider Table 3 when $\beta = 0.25$ and $\alpha = 0.25$, the respective ARL for ρ = **0.00, 0.25, 0.50** and **0.75** are **4.01, 3.98, 3.97** and **3.68**. This behavior is seen for all the combination of shifts in the location and dispersion parameters (α and β) of the process and, therefore, confirms the improvement on the sensitivity of the proposed control chart with the addition of the auxiliary variable (X). It equally indicates that as the correlation coefficient increases, the sensitivity of the proposed chart to OOC signals increases. When the performance of the proposed $M_R ELR_t$ chart is compared with the ELR_t and $M_d ELR_t$ charts, the following can be observed;

1. $M_R ELR_t$ chart has smaller OOC ARLs for all the correlation coefficients considered when compared with the ELR_t chart in [22] except at zero (0) correlation coefficient where both charts have almost the same performance. This claim can be observed in Tables 3-5. Therefore, $M_R ELR_t$ chart outperforms the ELR_t chart and the degree of this good performance increases with the increase in the correlation coefficient.
2. $M_R ELR_t$ chart and $M_d ELR_t$ chart have almost the same performance, and the use of any could be a matter of choice. However, this is only at small shifts in the location parameter as the $M_R ELR_t$ chart is seen to outperform the $M_d ELR_t$ chart when the shifts in the location parameter increases up to say, 1.00**.** This claim can also be observed in Tables 3-5. To be more specific, consider Table 5, from $\alpha = 1.00 - 1.50$, the OOC ARL of $M_R ELR_t$ chart is appreciably smaller than the corresponding values of the $M_d ELR_t$ chart in [15]. It is on the basis of this that we say that the $M_R ELR_t$ chart is superior to the $M_d ELR_t$ chart.
3. The ARLs for the $M_R ELR_t$ chart converges to zero (0) fastest with increased shifts in the IC parameters when compared to ELR_t and $M_d ELR_t$ charts.

From Table 1, it is clear that the chart identified all the first ten samples as IC samples, which they are. But, given the second ten OOC samples, the chart sampled only three before raising the OOC alarm from the 14^{th}sample. This behavour attests to the sensitivity of the proposed chart in detecting true OOC situation.

IV. Conclusion

The $M_R ELR_t$-EWMA chart has been found to be able to monitor for both the location and dispersion parameters concurrently and, as such, should be classified among the joint quality control charts. The $M_R ELR_t$-EWMA chart is simpler than many joint quality control charts both in structure and in application and, as such, can gain more popularity in usage than many other charts. It is even simpler than the $M_d ELR_t$-EWMA chart. This is because of the complexity of the $M_d ELR_t$-EWMA chart estimator compared to the $M_R ELR_t$-EWMA chart.

The results and discussion on $M_R ELR_t$-EWMA chart reveal that it is superior to the ELR_t-chart in detecting an OOC situation in all circumstances but only superior to $M_d ELR_t$-EWMA chart in detecting larger shifts from the IC process location. However, the performance of the $M_d ELR_t$-EWMA chart and $M_R ELR_t$-EWMA chart in detecting small shifts from the IC parameters (location and dispersion) is not better than each other and, therefore, their usage should be a matter of preference in that circumstance. With increased shifts, the ARL and MRL of the proposed $M_R ELR_t$-EWMA chart converges faster to zero (0) than the ELR_t- and $M_d ELR_t$-EWMA charts, which is also an indication of its better performance.

Conflict of Interests

The authors declare no conflict of interests

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