Soft Cushioned and Soft Star Refinements of Open Covers in Intuitionistic Fuzzy Soft Topological Spaces and Their Covering Dimension

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ABSTRACT. The concept of intuitionistic fuzzy soft topological spaces has emerged as a natural extension that integrates the ideas of intuitionistic fuzzy sets (IFS) and soft sets, providing a flexible framework for addressing complex problems characterized by uncertainty and imprecision. Within this context, open coverings and their refined forms such as cushioned refinements and star refinements play a crucial role in understanding the topological structure of spaces, particularly in relation to the notion of covering dimension. In this paper, we introduce the notions of intuitionistic fuzzy soft cushioned refinement, intuitionistic fuzzy soft star refinement, and intuitionistic fuzzy soft strongly star refinement of coverings within the framework of intuitionistic fuzzy soft topological spaces. These concepts are utilized to establish fundamental results concerning the covering dimension of intuitionistic fuzzy soft normal topological spaces. The obtained results contribute to a deeper understanding of covering properties in intuitionistic fuzzy soft topology and extend several classical topological concepts to the intuitionistic fuzzy soft framework.

Keywords. Intuitionistic fuzzy soft set, intuitionistic fuzzy soft topology, intuitionistic fuzzy soft star refinements, covering dimension.

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1. Introduction

Classical mathematical methods are often insufficient for addressing many real-world problems characterized by uncertainty, vagueness, and incomplete information. Consequently, several mathematical theories have been developed to overcome these limitations, including fuzzy set theory [23], intuitionistic fuzzy sets (IFS) [4], rough set theory [18], and soft set theory [16].These theories have found significant applications in topology and in various branches of mathematics. The topological structures based on fuzzy, soft, fuzzy soft, intuitionistic fuzzy, and intuitionistic fuzzy soft concepts were respectively introduced by Chang [7], Shabir and Naz [19], Tanay et al. [20], D. Coker [8], and Z. Li et al. [10]. Molodtsov et.al.[16] highlighted several important directions for the application of soft set theory, including smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory. These applications demonstrate the effectiveness of soft sets in modeling complex problems arising in engineering, computer science, economics, social sciences, and medical sciences. In particular, soft set theory provides a highly flexible mathematical framework through the incorporation of parameters. Maji et al. [11–14] introduced the concept of intuitionistic fuzzy soft sets, which generalizes both fuzzy soft sets [12] and classical soft sets [15], thereby offering a more powerful framework for dealing with uncertainty and imprecision.

In this paper, we introduce and develop the topological structures of intuitionistic fuzzy soft cushioned refinements, intuitionistic fuzzy soft star refinements, and intuitionistic fuzzy soft strongly star refinements of coverings. Furthermore, we define and investigate the covering dimension of intuitionistic fuzzy soft normal and paracompact topological spaces. In addition, we study the covering dimension of intuitionistic fuzzy soft normal paracompact spaces by employing star-refinement and strongly star-refinement techniques.

2.Preliminaries

In this section, we present and recall the basic definitions and concepts of intuitionistic fuzzy soft set (IFSS), and intuitionistic fuzzy soft topology (IFST).

Definition 2.1 [12] Let X be an initial universe set and E be the set of parameters. let IF^X denotes the collection of all intuitionistic fuzzy subsets of X. Let $A \subseteq E$, a pair **(f, A) or** f_A is called an intuitionistic fuzzy soft set over X. Where f is a mapping given

by $f: A \to IF^X$.

In general, for every e∈A, f(e) is an IFS set of X and it is called IF valueset of parameter e. Clearly, f(e) can be written as an IFS set such that

$$f(e) = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}.$$

The set of all IF soft sets over X with parameters from E is called an IF soft class and it is denoted by **IFS(X_E).**

Example 2.2 [17] Let (f, A) = f_A describe the character of the students with respect to the given parameters, for finding the best student of an academic year.Let the set of students under consideration be X ={ x_1, x_2, x_3, x_4}. Let E = good result (r), conduct (c), games and sports performances (g), sincerity(s), pleasing personality (p) be the set of parameters framed to choose the best student. Suppose Mr .Y has the parameter set A = { r, c, p } ⊆ E to choose the best student. Then f_A be the an IF soft set over *X*, defined as follows:

$$f(r) = \{(x_1, 0.8, 0.1), (x_2, 0.7, 0.5), (x_3, 0.9, 0.1), (x_4, 0.7, 0.2)\},$$

$$f(c) = \{(x_1, 0.6, 0.2), (x_2, 0.7, 0.1), (x_3, 0.5, 0.3), (x_4, 0.3, 0.6)\},$$

f(p) = {(x$_1$, 0.8, 0.1), (x$_2$, 0.7, 0.5), (x$_3$, 0.8, 0.1), (x$_4$, 0.7, 0.2)}}.In short we will represent f_A as:

f_A = {{(x$_1$, 0.8, 0.1), (x$_2$, 0.7, 0.05), (x$_3$, 0.9,0.1),(x$_4$,0.7,0.2)}, {(x$_1$, 0.6, 0.2), (x$_2$, 0.7, 0.1), (x$_3$, 0.5, 0.3)}, {(x$_4$, 0.3, 0.6)},{(x$_1$, 0.8, 0.1), (x$_2$, 0.7, 0.5), (x$_3$, 0.8, 0.1), (x$_4$, 0.7, 0.2)}},

Definition 2.3 (1). An IFS set (f, A) over X is said to be **absolute** intuitionistic fuzzysoft set denoted by $\widehat{A}$ if $\forall e \in A$, $f(e)$ is the **absolute** intuitionistic fuzzy soft set $\widehat{1}$ of X where $\widehat{1}(x) = 1$, $\forall$ x$\in$X.

We would use the notation $\widehat{X}_E$ to represent the absolute intuitionistic fuzzy soft set with respect to the set of parameters A.

(2). An IFS set (f, A) over X is said to be **null** intuitionistic fuzzysoft set denoted by $\tilde{\Phi}$ if

$\forall e \in A \subseteq E$, $f(e)$ is the null intuitionistic fuzzy set $\tilde{0}$ of X where $\tilde{0}(x) = 0$, $\forall$ x$\in$X.

We would use the notation $\tilde{\Phi}_A$ to represent the null intuitionistic fuzzysoft set with respect to the set of parameters A.

Definition 2.4 An IF soft set f_A is said to be **IF soft point over X**, denoted by e_f, if for the element e ∈ A, f(e) $\neq \widehat{0}$ and f (é) = $\widehat{0}$,

∀ é ∈ A − {e}.

Definition 2.5 A IF soft point e_f is said to be in IF soft set (g, A),denoted by $e_f \,\widehat{\in}\, (g, A)$ if for the element e ∈ A, f(e) ≤ g(e).

From now on, let IFSP(X) be the family of all intuitionistic fuzzy soft points over X.

Definition 2.6 Let A, B ∈ IFS(f).Then

1. (f, A) is called IFS **subset** of IFS set (g, B) (i.e. $(f, A) \,\widehat{\subseteq}\, (g, B)$) if A ⊆ B and f(e) ⊆ g(e), for all e∈A

2. (f, A) and (g, B) are called IF soft **equal**, if $(f, A) \,\widehat{\subset}\, (g, B)$ and $(f, B) \,\widehat{\subset}\, (g, A)$, we write f_A=g_B.

Obviously, f_A = g_B if and only if A = B and f (e) = g(e) for any e ∈ A.

The operations on IFTSS union, intersection, complement and others can be found in[5-6,11-13]

Definition 2.7 [10] Let $\tau \subset$ IFSS (X_E) be the collection of intuitionistic fuzzy soft sets over X, then τ is said to be an intuitionistic fuzzy soft topology on X if

1. $\widehat{\Phi}_E$, $\widehat{1}_E$ belong to τ,

2. If $(f, A), (g, B) \in \tau$, then $(f, A) \widehat{\cap} (g, B) \in \tau$

3. If $(f, A)_\lambda \in \tau \quad \forall \lambda \in \Lambda$, then $\widehat{\bigcup}_{\lambda \in \Lambda} (f, A)_\lambda \in \tau$

The binary (X_E, τ) or (X, τ, E) is called an intuitionistic fuzzy soft topological space over X (IFSTS). If $(f, E) \in \tau$, then the intuitionistic fuzzy soft sets (f, E) is said to be intuitionistic fuzzy soft open set, and its complement $(f, E)^c$ is closed.

Example 2.8 [9]. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $\tau = \{\widehat{\Phi}_E, \widehat{1}_E, f_{1E}, f_{2E}, f_{3E}, f_{4E}\}$, where $f_{1E}, f_{2E}, f_{3E}, f_{4E}$ are IF soft sets over X, defined as follows

$f_1(e_1)(x_1) = (0.2, 0.8), \quad f_1(e_1)(x_2) = (0.6, 0.3),$

$f_1(e_2)(x_1) = (0.2, 0.5), \quad f_1(e_2)(x_2) = (0.9, 0.1),$

$f_2(e_1)(x_1) = (0.1, 0.8), \quad f_2(e_1)(x_2) = (0.6, 0.1),$

$f_2(e_2)(x_1) = (0.2, 0.8), \quad f_2\, e_2)(x_2) = (0.8, 0.1),$

$f_3(e_1)(x_1) = (0.2, 0.8), \quad f_3(e_1)(x_2) = (0.6, 0.1),$

$f_3(e_2)(x_1) = (0.2, 0.5), \quad f_3(e_2)(x_2) = (0.9, 0.1),$

$f_4(e_1)(x_1) = (0.1, 0.8), \quad f_4(e_1)(x_2) = (0.6, 0.3),$

$f_4(e_2)(x_1) = (0.2, 0.8), \quad f_4(e_2)(x_2) = (0.8, 0.1).$

Then

$\tau = \{\tilde{\Phi}_E, \tilde{1}_E, (\{x(0.2,0.8), x(0.6,0.3)\}, \{x(0.2,0.5), x(0.9,0.1)\}), (\{x(0.1,0.8), x(0.6,0.1)\}, \{x(0.2,0.8), x(0.8,0.1)\}), (\{x(0.2,0.8), x(0.6,0.1)\}, \{x(0.2,0.5), x(0.9,0.1)\}),$

$(\{x(0.1,0.8), x(0.6,0.3)\}, \{x(0.2,0.8), x(0.8,0.1)\})\}$ is an IF soft topology on X and hence (X_E, τ) is an IF soft topological space over X.

Definition 2.9 Let X be an initial universe set, E be the set of parameters and $\tau = \{ \hat{\Phi}_E , \hat{1}_E \}$. Then τ is called the intuitionistic fuzzy soft indiscrete topology on X and (X_E,τ) is said to be a intuitionistic fuzzy soft indiscrete space over X.

Definition 2.10 [17] Let (X_E, τ) be an IF soft topological space over X and Y bea non-empty subset of X. Then

$$\tau_Y = \{(Y_f, E) : (f, E) \in T\}$$

is said to be the IF soft topology on Y and (Y_E, τ_Y) is called a IF soft subspace of (X_E, τ). Where $(Y_f, E) = Y_E \,\hat{\cap}\, (f, E)$

We can easily verify that τ_Y is in fact, an IF soft topology on Y.

Definition 2.11 [22] Let (X_E, τ) be an intuitionistic fuzzy soft topological space on X. An intuitionistic fuzzy soft set (f, E) over X is called an intuitionistic fuzzy soft **neighborhood** of an intuitionistic fuzzy soft point $e_f \in IFS(x_E)$ if there exists intuitionistic fuzzy soft open set (g, E) such that

$$e_f \,\hat{\in}\, (g, E) \,\hat{\subseteq}\, (f, E)$$

Definition 2.12 The intuitionistic fuzzy soft interior and intuitionistic fuzzy soft closure of an intuitionistic fuzzy soft set (f, A) in IFTS (X_E, τ) are denoted and defined respectively as follows:

IFSint$(f, A) = \hat{\bigcup}\, \{(g, B):(g, B)$ is an IFSOS in X and $(g, B) \,\hat{\subseteq}\, (f, A)\}$,

IFScl$(f, A) = \hat{\bigcap}\, \{(g, B):(g, B)$ is an IFSCS in X and $(f, A) \,\hat{\subseteq}\, (g, B)\}$

Definition 2.13 let (X_E, τ) be an IF soft topological space over X, (F, E) and (G, E)IF soft closed sets over X such that $(F, E) \,\hat{\cap}\, (G, E) = \hat{\Phi}_E$. If there exist IF soft open sets (U_1, E) and (U_2, E) such that

$(F, E) \,\hat{\subseteq}\, (U_1, E), (G, E) \,\tilde{\subseteq}\, (U_2, E)$, and $(U_1, E) \,\hat{\cap}\, (U_2, E) = \hat{\Phi}_E$, then (X_E, τ) is called an IF soft **normal** space.

Equivalently

(X_E, τ) is IF soft **normal** if for each IFS closed set (F, E) and each IFS open set (U, E) in X with $(F, E) \,\hat{\subseteq}\, (U, E)$ there exist IF soft open (V,E) such that $(F,E) \,\hat{\subseteq}\, (V,E) \,\hat{\subseteq}\,$ IFSCl$(V,E) \,\hat{\subseteq}\, (U,E)$.

Definition 2.14 A family $\Psi = \{(f_i, E) : i \in I\}$ of IF soft sets is an IFS **open cover** of IF soft set (f, A) if $(f, A) \widehat{\subseteq} \widehat{\bigcup_{i \in I}} \{(f_i, E) : (f_i, E) \widehat{\in} \Psi\}$. A sub cover of Ψ is a subfamily of Ψ, which is also IFS open cover.

According to the definition (2-7) and [3], we give the following definition:

Definition 2.15 The family $\Psi = \{(f_i, E) : i \in I\}$ is an IFS covering of $\tilde{1}_E$ if and only if $\widehat{\bigcup_{i \in I}} \{(f_i, E)\} = \widehat{1}_E$, and we say that Ψ is a covering of IFSS X_E, and a collection $\xi = \{(g_\lambda, E) : \lambda \in \Gamma\}$ is said to be refinement of Ψ if $\widehat{\bigcup_{\lambda \in \Lambda}} \{(g_\lambda, E)\} = \widehat{1}_E$, and each (g_λ, E) is contained in some members (f_i, E) of Ψ.

Definition 2.16 [19] Let (X_E, τ) be an intuitionistic fuzzy soft topological space.

A family ξ of intuitionistic fuzzy soft sets over X is called (X_E, τ) **locally finite** if for each IF soft point $e_f \in$ IFSP(X), there exists intuitionistic fuzzy soft neighborhood (f, E) over X that intersects only finitely many elements of ξ.

Definition 2.17 [19] An intuitionistic fuzzy soft topological space (X_E, τ) is called intuitionistic fuzzy soft **paracompact** space if each intuitionistic fuzzy soft open covering Ψ of $\tilde{1}_E$ has an intuitionistic fuzzy soft open locally finite refinement ξ that covers $\tilde{1}_E$.

Definition 2.18. We say that IFSTS (X_E, τ) is paracompact and normal if its normal and each intuitionistic fuzzy soft open covering of $\tilde{1}_E$ has an intuitionistic has an intuitionistic fuzzy soft open locally finite refinement.

Definition 2.19 Let (X_E, τ) be IF soft topological space and $(F, E) \in$ IFS(X_E).Then the IF soft set (F, E) is called **compact** if each IF soft open cover of (F, E) has a finite sub cover. Also IF soft topological space (X_E, τ) is called compact if each IFS open cover of $\widehat{1}_E$ has a finite sub cover.

Definition 2.20 [1] Let X be a nonempty set. A family $\Psi = \{(f_i, E) : i \in I\}$ of IFS sets in X is said to be of order n $(n > -1)$ written $ord_{IFS}\Psi = n$, if n is largest integer such that there exists an overlapping subfamily of Ψ having n + 1 elements.

Remark 2.21 [1-2] From the above definition if $ord_{IFS}\boldsymbol{\Psi} = n$ then for each n + 2 distinct indexes $i_1, i_2, \ldots, i_{n+2} \in I$,we have $(f_1, E)_{i_1} \widehat{\bigcap} (f_2, E)_{i_2} \widehat{\bigcap} \ldots \widehat{\bigcap} (f_n, E)_{i_{n+2}} = \widehat{\Phi}_E$. Then it is non-overlapping, in particular if $\mathrm{ord}_{IFS}\Psi = -1$, then Ψ consists of the empty IFS sets and $\mathrm{ord}_{IFS}\Psi = 0$, then Ψ consist of pair wise disjoint IFS sets which are not all empty.

Definition 2.22 The covering dimension of a IFST (X_E, τ) denoted $\dim_{IFS}(X_E)$ is the least integer n such that every finite open cover of $\widehat{1}_E$ has a finite open refinement of order not exceeding n or $+\infty$ if there exists no such integer.
Thus it follows that $\dim_{IFS}(X_E) = -1$ if and only if $X = \emptyset$ and $\dim_{IFS}(X_E) \leq n$ if every finite open cover of $\tilde{1}_E$ has a finite open refinement of order $\leq n$. We have $\dim_{IFS}(X_E) = n$ if it is true that $\dim_{IFS}(X_E) \leq n$, but it is false that $\dim_{IFS}(X_E) \leq n - 1$. Finally $\dim_{IFS}(X_E) = +\infty$ if for every positive integer n it is false that $\dim_{IFS}(X_E) \leq n$.

Remark 2.23 [3] The notion of covering dimension of IFST (X_E, τ) is intuitionistic fuzzy topological invariant. Moreover, the covering dimension of a topological space is n if and only if the covering dimension of its characteristic intuitionistic soft topological space is n.

3. Intuitionistic fuzzy soft cushioned refinement of IFS open Covering.

Now, we give the following definition for **cushioned** refinement of ***intuitionistic* fuzzy soft** open covering:

Definition 3.1 Let (X_E, τ) be an IFST. An intuitionistic fuzzy soft cushioned refinement of an IFS open covering $\Psi = \{(f_i, E): i \in I\}$ of $\widehat{1}_E$ is an IFS closed covering $\xi = \{(g_i, E): i \in I\}$ of $\widehat{1}_E$ such that for each sub set J of I,
$\widehat{\bigcup_{i \in J}}\{\overline{(g_i, E)} \widehat{\subseteq} \widehat{\bigcup_{i \in J}}(f_i, E)\}$

Proposition 3.2 If IFST (X_E, τ) is intuitionistic fuzzy soft paracompact and normal spaces then each *intuitionistic fuzzy* soft open covering $\widehat{1}_E$ **has** *intuitionistic fuzzy* soft **cushioned** refinement.

Proof. If $\Psi = \{(f_\lambda, E): \lambda \in \Lambda\}$ is an intuitionistic fuzzy soft open covering of a paracompact normal space (X_E, τ), i.e

$$\widehat{\bigcup_{\lambda \in \Lambda}}\{(f_\lambda, E)\} = \widehat{1}_E$$

Then there exists a locally finite intuitionistic fuzzy soft closed covering $\xi = \{(g_\lambda, E): \lambda \in \Lambda\}$ of $\widehat{1}_E$ such that $(g_\lambda, E) \widehat{\subseteq} (f_\lambda, E)$ for each $\lambda \in \Lambda$. The intuitionistic fuzzy soft covering ξ is evidently a cushioned refinement of Ψ. ∎

Proposition 3.3 If IFST (X_E, τ) is an intuitionistic fuzzy soft paracompact and normal spaces and $T : (X_E, \tau) \rightarrow (Y_E, \tau)$ is an intuitionistic fuzzy soft closed surjection then (Y_E, τ) is an intuitionistic fuzzy soft paracompact and normal.

Proof Let $\Psi = \{(f_i, E) : i \in I\}$ is an intuitionistic fuzzy soft open covering of (Y_E, τ).Then $T^{-1}(\Psi) = \{T^{-1}(f_{i,}E), i\epsilon I\}$ is an intuitionistic fuzzy soft open covering of (X_E, τ) so that by Proposition 3.2 there exists an intuitionistic fuzzy soft closed covering $\xi = \{(g_i, E) : i \in I\}$ of $\widehat{1}_E$ which is intuitionistic fuzzy soft cushioned refinement of $T^{-1}(\Psi)$. Since T is an *intuitionistic* fuzzy soft closed surjection, $T(\xi) = \{T(g_i, E)\}_{i\in I}$is an intuitionistic fuzzy soft closed covering of (Y_E, τ).
Moreover $\overline{T((f,E))} = T(\overline{(f,E)})$ for each intuitionistic fuzzy soft set in (X_E, τ).Thus if μ is a subset of I, then since $\overline{(g_i,E)_{i\in I}} \widehat{\subseteq} \widehat{\bigcup}_{i\in\mu} T^{-1}(f_i,E) = T^{-1}(\widehat{\bigcup}_{i\in\mu}(f_i,E))$,it follows that

$$\overline{\widehat{\bigcup}_{i\in\mu} T(g_i,E)_{i\in I}} = \overline{(T(\widehat{\bigcup}_{i\in\mu}(g_i,E))} = T(\overline{\widehat{\bigcup}_{i\in\mu}(g_i,E)} \widehat{\subseteq} \widehat{\bigcup}_{i\in\mu}(f_i,E)$$

Thus ψ has an intuitionistic fuzzy soft cushioned refinement and it follows from Proposition 3.2 that (Y_E, τ) is an intuitionistic *fuzzy* soft paracompact normal space.■

4. *Intuitionistic fuzzy soft* star refinement of IFS open Covering.

Definition 4.1 Let $\xi = \{(g_\lambda, E) : \lambda \in \Lambda\}$ be an intuitionistic fuzzy soft family of subsets of IFSTS (X_E, τ) and (f, E) be an intuitionistic fuzzy soft set in (X_E, τ) then st((*f*,E),ξ) is the union of those sets (g_λ, E) such that $(g_\lambda, E) \widehat{\bigcap} (f, E) \neq \widehat{\Phi}_E$, and if e_f is an intuitionistic fuzzy soft point of $\widehat{1}_E$, we write $st(e_g, \xi)$ to denote the union of those sets (g_λ, E)such that $e_g \widehat{\in} (g_\lambda, E)$.

Now, we give the following definition for **star** refinement and strongly star-refinement of ***intuitionisti*c fuzzy soft** open covering:

Definition 4.2 Let (X_E, τ) be IFST and $\xi = \{(g_\lambda, E) : \lambda \in \Lambda\}$,
$\Psi = \{(f_i, E) : i \in I\}$ be two intuitionistic fuzzy soft open coverings of $\widehat{1}_E$ we say ξ is an intuitionistic fuzzy soft **star refinement** of Ψ if the covering $st(e_g, (g_\lambda, E))$ is a refinement of Ψ. where
$st(e_g, (g_\lambda, E) = \bigcup_{\lambda\in\Lambda} \{(g_\lambda, E) \widehat{\in} \xi : e_f \widehat{\in} (f_\lambda, E)\}$. i.e. $(g_\lambda, E) \subset (f_i, E)$ for some$\lambda \in \Lambda$and for each $i \in I$.
Also we say that ξ is intuitionistic fuzzy soft **strongly star** – refinement of Ψ if the covering st((g,E),ξ) is a refinement of Ψ. i.e. every intuitionistic fuzzy soft open set

(g,E) $\in\xi$ there exist an intuitionistic fuzzy soft set (g,E)$\in\Psi$ such that st((g,E),ξ) $\widehat{\subseteq}$ Ψ, where st((g,E),ξ) denotes the star of the intuitionistic fuzzy soft set ξ with respect to the intuitionistic fuzzy soft cover Ψ. i.e.

st((g,E),ξ) $= \widehat{\bigcup_{\lambda\in\Lambda}}\{(g,E)\widehat{\in}\xi:((g,E)\widehat{\bigcap}((g_{\lambda},E)\neq\widehat{\Phi}_E\}$, for some $\lambda\in\Lambda$ and for each i$\in$I.

It is easy to verify the following lemma.

Lemma 4.3 Every finite intuitionistic fuzzy soft open cover of intuitionistic fuzzy soft paracompact normal topological spaces (X_E,τ) has a finite intuitionistic fuzzy soft open star refinement.

Proof. Let (X_E,τ) be an intuitionistic fuzzy soft paracompact normal space and let $\Psi=\{(f_i,E):i=1,2,...k,i\in I\}$ be an IFS open covering of $\widehat{1}_E$ .There exists an *intuitionistic fuzzy* soft closed covering $\Omega=\{(F_i,E):i=1,2,\dots k,i\in I\}$ of $\widehat{1}_E$ and an of *intuitionistic fuzzy* soft locally finite open covering $\zeta=\{(V_i,E):i=1,2,...k,i\in I\}$ such that $(F_i,E)\widehat{\subseteq}(V_i,E)\widehat{\subseteq}(f_i,E)$ for each i. For each intuitionistic fuzzy soft point $e_f\in IFS(x_E)$, $e_f\widehat{\in}\widehat{1}_E$, choose an intuitionistic fuzzy soft open neighborhood (V, E) of e_f such that

$I_{e_f}=\{i\in I:(V_{e_f},E)\widetilde{\bigcap}(V_i,E)\neq\widehat{\Phi}_E\}$ is finite. Let

$I_{e_f}^{\wedge}=\{i\in I_{e_f}:e_f\widehat{\in}(V_i,E)\}$ and $I_{e_f}^{\wedge\wedge}=\{i\in I_{e_f}:e_f\widehat{\notin}(F_i,E)\}$.

It's clear that $I_{e_f}^{\wedge}\widehat{\bigcap}I_{e_f}^{\wedge\wedge}=I_{e_f}$ .Now, for each e_f in X, let

$W_{e_f}=\{(V_{e_f},E)\widehat{\bigcap}(\widehat{\bigcap_{i\in I_{e_f}^{\wedge}}}(V_i,E))\widehat{\bigcap}(\widehat{\bigcap_{i\in I_{e_f}^{\wedge\wedge}}}(F_i,E)^c)\}$.Then $\Theta=\{(W_{e_f},E)\}$ is an IFS open covering of $\widehat{1}_E$. If e_f intuitionistic fuzzy soft point in $\widehat{1}_E$, there exists i_0 in I such that e_f $\widehat{\in}$ (F_{i_0}, E). If $e_f\in(W_{e_g},E)$, then $(W_{e_g},E)\widehat{\bigcap}(F_{i_0},E)\neq\widehat{\Phi}_E$ so that $i_0\in I_{e_g}$ and $i_0\notin I_{e_g}^{\wedge\wedge}$ Thus $i_0\widehat{\notin}I_{e_g}^{\wedge}$ so that $(W_{e_g},E)\widehat{\subseteq}(V_{i_0},E)$. Thus

$st(e_g,(\Theta,E)\widehat{\subseteq}(V_{i_0},E)\widehat{\subseteq}(f_{i_0},E)$

Hence Θ is an intuitionistic fuzzy soft star-refinement of Ψ . ■

Proposition 4.4 A normal IFS topological space (X_E,τ) satisfies $\dim_{IFS}(X_E)\leq n$ if each IFS finite open covering of $\tilde{1}_E$ has a IFS star finite open refinement which is IFS finite open covering of $\tilde{1}_E$ of order $\leq$ n.

Proof. Suppose $\dim_{IFS}(X_E) \leq n$, for a normal IFST (X_E, τ)), let $\Psi = \{(f_i, E) : i = 1, 2, \ldots k, i \in I\}$ be an IFS open covering of $\tilde{1}_E$,and since (X_E, τ) is IFS normal, there exists an IFS closed cover $\Omega = \{(F_i, E) : i = 1, 2, \ldots k, i \in I\}$ of $\tilde{1}_E$ such that $(F_i, E) \,\tilde{\subseteq}\, (f_i, E)$ then $(F_i, E) \,\tilde{\subseteq}\, (f_i, E) \,\tilde{\subseteq}\, \tilde{1}_E$ which implies that

$\tilde{1}_E - (F_i, E) = (F_i, E)^c$. Let Δ be the set of non-empty subset of {1,2,…,k} . For each δ in Δ let $\xi_\delta = (\tilde{\bigcap}_{i \in \delta}(f_i, E) \tilde{\bigcap} (\tilde{\bigcap}_{i \notin \delta}(\tilde{1}_E - (F_i, E))$ where $\delta = \{i : e_f \,\tilde{\subseteq}\, (f_i, e)\}$.

Now if e_f is an IFS point in X i.e. $e_f \,\tilde{\subseteq}\, 1_{\tilde{E}}$. Then $e_f \,\tilde{\subseteq}\, \xi_\delta$ .since $e_f \,\tilde{\subseteq}\, \tilde{\bigcup}\xi_\delta$ if and only if $\exists\, \delta \in \Delta, e_f \,\tilde{\subseteq}\, \xi_\delta$. Thus $\zeta = \{\xi_\delta : \delta \in \Delta\}$ is an IFS finite open cover of $\tilde{1}_E$ furthermore, ζ is star- refinement of Ψ. For if e_f is an IFS point in $\tilde{1}_E$ then $e_f \,\tilde{\subseteq}\, (F_j, E)$ for some j, since $\tilde{1}_E = \tilde{\bigcup}_{j \in \Delta}(F_j, E)$ and if $e_f \,\tilde{\subseteq}\, \xi_\delta$ then $V_\delta \tilde{\bigcap} F_j \neq \mathbf{\Phi}^{\sim}$ thus j∈ δ, where $\delta = \{j : e_f \,\tilde{\subseteq}\, \xi_j\}$ and since $(F_j, E) \,\tilde{\subseteq}\, (f_j, E)$, (X_E, τ) is an IFS normal there is ζ_δ in **X** such that $e_f \,\tilde{\in}\, (F_j, E) \,\tilde{\subseteq}\, (\xi_\sigma, E) \,\tilde{\subseteq}\, \overline{(\xi_\sigma, E)} \,\tilde{\subseteq}\, (f_j, E)$ so that $\zeta_\delta \,\tilde{\subseteq}\, (f_j, E)$. Thus $st(e_f, \zeta) = \tilde{\bigcup}\{(\xi_\delta, E) \,\tilde{\in}\, \xi : e_f \,\tilde{\in}\, (\xi_\delta, E)\} \,\tilde{\subseteq}\, (f_j, E)$. Hence $st(e_f, \zeta) \,\tilde{\subseteq}\, (f_j, E)$.

Now, since $\dim_{IFS}(X_E) \leq n$, there exists an IFS finite open refinement Ξ of Ω such that the order of Ξ does not exceed n, since Ω is star- refinement of Ψ, it follows that Ξ is star- refinement of Ψ. ■

Proposition 4.5 If (X_E, τ) is an IFS paracompact normal space satisfies $\dim_{IFS}(X_E) \leq n$, and $\{(\Omega_i, E) : i = 1, 2, \ldots k, i \in I\}$ be a sequence of an IFS open covering of $\tilde{1}_E$,there exist a sequence $\{(\Psi_i, E) : i = 1, 2, \ldots k, i \in I\}$ of an IFS open covering of $\tilde{1}_E$ of order ≤ n, such that for each i, (Ψ_i, E) is an IFS refinement of (Ω_i, E), and (Ψ_{i+1}, E) is an IFS strong star- refinement of (Ψ_i, E) and each member of (Ψ_{i+1}, E) meets at most n + 1 members of (Ψ_i, E).

Proof. Since (X_E, τ) is an IFS paracompact normal space such that $\dim_{IFS}(X_E) \leq n$, there exists IFS locally finite open refinement (Ψ_1, E) of (Ω_1, E) such that the order of $\{\overline{f, E}) : (f, E) \,\tilde{\in}\, (\Psi_1, E)\}$ does not exceed n. Suppose that (Ψ_i, E) is an IFS locally finite open covering of $\tilde{1}_E$ such that the order of $\{\overline{f, E}) : (f, E) \,\tilde{\in}\, (\Psi_i, E)\}$ does not exceed n. If e_f is an intuitionistic fuzzy soft point of $\tilde{1}_E$, let

$G_{e_f} = \hat{1}_E - \cup\left\{\left\{\overline{(f,E)} : (f,E) \hat{\in} (\Psi_i, E)\right\} and \quad e_f \hat{\notin} \overline{(f,E)}\right\}$, and let

$(\Theta_{i+1}, E) = \{G_{e_f}\}$. Then by Proposition (4.4) there exists an IFS open strong star-refinement (Ξ_{i+1}, E) of (Ψ_i, E). Now, since (X_E, τ) is an IFS paracompact normal space such that $\dim_{IFS}(X_E) \leq n$, we can choose IFS locally finite open covering (Ψ_{i+1}, E) of $\hat{1}_E$, which is an IFS refinement of the all IFS open coverings $(\Theta_{k+1}, E), (\Omega_{i+1}, E) \quad and\ (\Xi_{i+1}, E)$ such that the order of $\left\{\overline{(f,E)} : (f,E) \hat{\in} (\Psi_{i+1}, E)\right\}$ does not exceed n. Clearly (Ψ_{i+1}, E) is an IFS strong star-refinement of (Ψ_i, E). If $(f,E) \hat{\in} (\Psi_{k+1}, E)$ then $(f,E) \hat{\subseteq} \mathbf{G}e_f$ for some e_f in $\hat{1}_E$. Thus if $(V,E) \hat{\in} (\Psi_k, E)$ and $(f, E) \hat{\cap} (V, E) \neq \hat{\Phi}_E$, then $e_f \hat{\in} \overline{(V,E)}$. Since the order of $\left\{\overline{(V,E)} : (V,E) \hat{\in} (\Psi_i, E)\right\}$ does not exceed n, it follows that (f, E) meets at most n + 1 members of (Ψ_i, E). It now follows by induction that we can construct the required sequence $\{(\Psi_i, E) : i = 1, 2, \ldots k, i \in I\}$ of an IFS open coverings of $\hat{1}_E$.■

5. Conclusion

In this paper, we introduced intuitionistic fuzzy soft cushioned refinements, star refinements, and strongly star refinements in intuitionistic fuzzy soft topological spaces. Using these concepts, we studied the covering dimension of intuitionistic fuzzy soft normal and paracompact spaces and obtained several results extending classical dimension theory to the intuitionistic fuzzy soft setting. In future work, we plan to investigate additional dimension-type properties in intuitionistic fuzzy soft topological spaces, including inductive dimensions and large-scale covering properties, as well as exploring possible applications in decision-making and intelligent systems.

Conflict of interest.

No potential conflict of interest relevant to this article was reported.

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