# Numerical Solution Of First Order Fuzzy Differential Equation By Fourth Order Runge Kutta Method Based On Linear Combination Of Arithmetic Mean, Harmonic Mean And Geometric Mean

Dr.R.Gethsi Sharmila [*1], S.Suvitha [*2]

[#] *Dept., of Mathematics, Bishop Heber College (Autonomous),*

*Abstract* — **We propose fuzzy version of runge – kutta method of order four, based on linear combination of arithmetic mean, harmonic mean and geometric mean for the solution of first order linear fuzzy differential equations which is done by not converting them to crisp form. By solving a fuzzy initial value problem with trapezoidal fuzzy number, the efficiency and accuracy of the proposed method is illustrated.**

*Keywords*— **Fuzzy number, Trapezoidal fuzzy number, Fuzzy Differential Equations, RK method, Higher order derivatives etc.**

## I. Introduction

Fuzzy Differential Equations (FDEs) have a wide range of applications in many branches of engineering and medicine. Chang S.L. and Zadeh L.A. in [8] first introduced the concept of fuzzy derivative. It was followed up by Dubosis.D and Prade[9].who used extension principle in their approach. The term "fuzzy differential Equation" was introduced in1987 by Kandel.A and Byatt.W.J[14]. They have given many suggestions for the definition of fuzzy derivative to study "fuzzy differential Equation". In the litreture, there are several approaches to study fuzzy differential equations. The first and most popular one is Hukuhara derivative made by Puri.M.L Ralesu.D.A [18]. Seikkala [19] rigorously treated the FDE and the initial value problem (Cauchy problem). Here the solution of fuzzy differential equation becomes fuzzier as time goes on. This approach does not reproduce the rich and varied behaviour of ordinary differential equations. Bede.B and Gal.S.G [6,7] have introduced another concept of derivatives called the generalized Hukukara derivative. Under this interpretation, the solution of a fuzzy differential equation becomes less fuzzier as time goes on. A strong generalized derivative is defined for a large class of fuzzy number valued function than the Hukuhara derivative. This case of a fuzzy differential equation is not unique. it has two solution locally. The analytical methods for solving Nth - order linear differential equations with fuzzy initial value conditions was introduced by Buckley and Feuring [4]. Recently some Mathematicians have studied fuzzy differential equations by Numerical methods. Numerical Solution of fuzzy differential equations has been introduced by M.Ma, M.Friedman, and Kandel.A in [16] through Euler method. Taylors method and Runge –Kutta methods have also been studied by authors in [1],[2]. The existence of solutions for fuzzy differential equations have been extensively studied by several other authors in [3,4]. It is difficult to obtain an exact solution for fuzzy differential equations and hence several numerical methods were proposed in [10,11,12,17].

In this paper, a new numerical method to solve first order linear fuzzy differential equations is presented using the fourth order runge-kutta method based on linear combination of arithmetic mean, harmonic mean and geometric mean. The paper is organized as, In Section 2, some basic results on fuzzy numbers and fuzzy derivative are given. In Section 3, the fuzzy initial value problem is treated using the extension principle of Zadeh and the concept of fuzzy derivative. In Section 4, a fourth order runge-kutta method based on linear combination of arithmetic mean, harmonic mean and geometric mean is introduced. In Section 5 the proposed method is illustrated by solving two examples, and the conclusion is drawn in Section 6.

## II. PRELIMINARIES

***Definition 2.1***

A Fuzzy Number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values. Where each possible value has its own weight between 0 & 1. This weight is called the membership function.

1. A fuzzy number is a quantity. (i.e) is expressed as a fuzzy set defining a fuzzy interval in the real number R.

2. Convex fuzzy set

$$u(tx+(1-t)y) \geq \min\{u(x),u(y)\} \forall t \in [0,1], x, y \in R$$
$$u(tx+(1-t)y) \geq \min\{u(x),u(y)\} \forall t \in [0,1], x, y \in R$$

3. Normalized fuzzy set

$$\exists x_0 \in R \text{ with } u(x_0)=1$$

4. Its membership function is piecewise continuous of bounded support. We represent an arbitrary fuzzy number by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right), 0 \leq r \leq 1$ that satisfies the following requirements.

   1) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1],with respect to any r.

   2) $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0,1] with respect to any r

   3) $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$ then,

   The r-level set is $[u]_r = \{x / u(x) \geq r\}, 0 \leq r \leq 1$ is a closed and bounded interval, denoted by

   $[u]_r = \{\underline{u}(r) \leq \overline{u}(r)\}$ and clearly $[u]_0 = \{x / u(x) > 0\}$ is compact.

***Definition 2.2***

A trapezoidal fuzzy number $u$ is defined by four real number of the trapezoidal is the interval $[a, d]$ and its vertices at $x = b, x = c$ . Trapezoidal fuzzy number will be written as $u = (a,b, c, d)$ is defined as follows

$$\begin{cases} \dfrac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \leq c \\ \dfrac{x-d}{c-d} & c \leq x \leq d \end{cases}$$

We will have

(1) $u > 0 if a > 0$
(2) $u > 0 if b > 0$
(3) $u > 0 if c > 0$
(4) $u > 0 if d > 0$

***Definition 2.2.1***

A Trapezoidal fuzzy number $A = (a, b, c, d)$ is said to be zero trapezoidal fuzzy number if $a = 0, b = 0, c = 0, d = 0.$

***Definition 2.2.2***

A Trapezoidal fuzzy number $A = (a, b, c, d)$ is said to be non - negative trapezoidal fuzzy number if $a \geq 0$ .

***Definition 2.2.3***

A Trapezoidal fuzzy number $A = (a, b, c, d)$ and $B = (a_2, b_2, c_2, d_2)$ is said to be equal trapezoidal fuzzy number if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

## III. RUNGE – KUTTA METHOD FOR INITIAL VALUE PROBLEM

Consider the initial value problem

$$y'(t) = f(t, y(t))\ ;\ a \leq t \leq b \qquad (3.1)$$
$$y(a) = \alpha\ ,$$

The basis of all Runge-Kutta method is to express the difference between the value of y at $t_{n+1}$ and $t_n$ as

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i \qquad (3.2)$$

where for i=1, 2, …,m, $w_i$'s are constants and $k_i = h f\left(t_n + c_i h, y_n + \sum_{j=1}^{i-1} a_{ij} k_j\right)$.

Equations (3.2) is to be exact for powers of h through $h^m$ , because it is to be coincident with Taylor series of order m.

### *3.1 Fuzzy Initial Value Problem*

Consider the fuzzy initial value problem

$$y'(t) = f(t, y(t))\ ;\ 0 \leq t \leq T \qquad (3.3)$$
$$y(0) = y_0\ ,$$

with the grid points $0 \leq t_1 \leq t_2 \leq ... \leq t_N = T$ and $h = \frac{(b - a)}{N} = t_{i+1} - t_i$ (3.4)

where f is a continuous mapping from $R_+ \times R$ into R and $y_0 \in E^1$ with r-level sets

$$[y_0]_r = \left[\underline{y}(0;r), \overline{y}(0;r)\right], \ r \in (0,1],$$

The extension principle of Zadeh leads to the following definition of $f(t, y)$ when y = y(t) is a fuzzy number

$$f(t, y)\,(s) = \mathrm{Sup}\{y(\tau) \setminus s = f(t,\tau)\}, \ s \in R$$

It follows that

$$[f(t, y)]_r = [\underline{f}(t, y; r), \ \overline{f}(t, y; r)], \ r \in (0,1],$$

where

$$\underline{f}(t, y; r) = \min\{f(t, u) \setminus u \in [\underline{y}(r), \overline{y}(r)]\}$$
$$\overline{f}(t, y; r) = \max\{f(t, u) \setminus u \in [\underline{y}(r), \overline{y}(r)]\} \quad (3.5)$$

Then $y : [0, \infty) \to E^1$ is a solution of (3.1) using the seikkala derivative and $y_0 \in E^1$ if

$$\underline{y}'(t, y; r) = \min\{f(t, u) \setminus u \in [\underline{y}(r), \overline{y}(r)]\}, \ \underline{y}(0; r) = \underline{y_0}(r)$$
$$\overline{y}'(t, y; r) = \max\{f(t, u) \setminus u \in [\underline{y}(r), \overline{y}(r)]\}, \ \overline{y}(0; r) = \overline{y_0}(r)$$

for all $t \in [0, \infty)$ and $\alpha \in [0,1]$,.

***3.2 Theorem***

Let f satisfy

$$\left|f(t, v) - f(t, \bar{v})\right| \le g(t, \left|v - \bar{v}\right|), \ t \ge 0, \ v, \ \bar{v} \in R.$$

where g : $R_+ \times R_+ \to R_+$ is a continuous mapping such that r ⟶ g(t, r) is non-decreasing and the initial value problem

$$u'(t) = g(t, u(t)), u(0) = u_0. \quad (3.6)$$

has a solution on $R_+$ for $u_0 > 0$ and that u(t) = 0 is the only solution of (3.4) for $u_0 = 0$. Then the fuzzy initial value problem (3.1) has a unique fuzzy solution. Refer [26].

## IV. The Fourth Order Runge-Kutta Method Based on Linear Combination of Arithmetic Mean, Harmonic Mean and Geometric Mean for First Order Differential Equation.

A new fourh order Runge - Kutta method based on linear combination of arithmetic mean, harmonic mean and geometric mean for first order differential equation was proposed by using Khattri [15] as follows

$$RM(k_1, k_2) = \frac{14\,AM(k_1, k_2) - HM(k_1, k_2) + 32GM(k_1, k_2)}{45}$$

For the IVP of the form $y' = f(t, y)$, the Fourth order Runge - Kutta methods ([2] and [4]) using variety of means can be written in the form

$$y_{n+1} = y_n + \frac{h}{2}\left[\sum_{i=1}^{3} Means\right] \quad (4.1)$$

where means includes Arithmetic Mean(AM), Geometric Mean (GM), Contraharmonic Mean (CoM), Centroidal Mean (CeM), Root Mean Square (RM), Harmonic Mean(HaM), and Heronian Mean (HeM) which involves $k_{i,}\ 1 \le i \le 4$ , where,

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + a_1 h, y_n + a_1 h k_1)$$
$$k_3 = f(t_n + (a_2 + a_3) h, y_n + a_2 h k_1 + a_3 h k_2)$$
$$k_4 = f(t_n + (a_3 + a_4) h, y_n + a_2 h k_1 + a_3 h k_2 + a_4 h k_3)$$

Substituting the Arithmetic mean, Harmonic mean and Geometric mean, Bazuaye Frank [ 5] proposed 4th order Runge-Kutta method based on linear combination of arithmetic mean, harmonic mean and Geometric mean as follows.

$$y_{n+1} = y_n + \frac{h}{135}\left[7(k_1 + 2k_2 + 2k_3 + k_4) - \left(\frac{3k_1k_2}{k_1+k_2} + \frac{3k_2k_3}{k_2+k_3} + \frac{3k_3k_4}{k_3+k_4}\right) + 48\left(\sqrt{k_1+k_2} + \sqrt{k_2+k_3} + \sqrt{k_3+k_4}\right)\right]$$

*where*

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \frac{h}{4}, y_n + \frac{h}{4}k_1\right)$$
$$k_3 = f\left(x_n + \frac{h}{4}, y_n + \frac{h}{32}(-k_1 + 9k_2)\right)$$
$$k_4 = f\left(x_n + h, y_n + \frac{h}{96}(-6k_1 + 10k_2 + 44k_3)\right)$$

with the grid points $a = t_0 \le t_1 \le \ldots \le t_N = b$ and $h = \frac{(b-a)}{N} = t_{i+1} - t_i$ (4.2)

***4.1 Fourth Order RK Method for solving first order Fuzzy Differential Equations.***

Let us consider the first order fuzzy ordinary differential equations of the form

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases} \quad (1)$$

The equations (2) and (3) are exact and approximate solutions of equation (1)respectively

$$[Y(t_n)]_r = \left[\underline{Y}(t_n;r), \overline{Y}(t_n;r)\right] \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (2)$$
$$[y(t_n)]_r = \left[\underline{y}(t_n;r), \overline{y}(t_n;r)\right] \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (3)$$

By using Fourth order Runge-Kutta method the approximate solution is obtained as follows

$$\underline{y}(t_{n+r};r) = \underline{y}(t_n;r) + \sum_{j=1}^{4} w_j k_{j,1}(t_n, y(t_n;r))$$
$$\overline{y}(t_{n+r};r) = \overline{y}(t_n;r) + \sum_{j=1}^{4} w_j k_{j,2}(t_n, y(t_n;r)) \qquad (4.3)$$

Where the $w_j$'s are constants. Then $k_{j,1}$ and $k_{j,2}$ for j = 1,2,3,4 are defined as follows

$$k_{1,1}(t_n, y(t_n;r)) = \min h\left\{ f(t_n,u) / u \in (\underline{y}(t_n;r), \overline{y}(t_n;r) \right\}$$
$$k_{1,2}(t_n, y(t_n;r)) = \max h\left\{ f(t_n,u) / u \in (\underline{y}(t_n;r), \overline{y}(t_n;r) \right\}$$
$$k_{2,1}(t_n, y(t_n;r)) = \min h\left\{ f(t_n + \frac{h}{4}, u) / u \in (p_{1,1}(t_n, y(t_n;r)), p_{1,2}(t_n, y(t_n;r)) \right\}$$
$$k_{2,2}(t_n, y(t_n;r)) = \max h\left\{ f(t_n + \frac{h}{4}, u) / u \in (p_{1,1}(t_n, y(t_n;r)), p_{1,2}(t_n, y(t_n;r)) \right\}$$
$$k_{3,1}(t_n, y(t_n;r)) = \min h\left\{ f(t_n + \frac{h}{4}, u) / u \in (p_{2,1}(t_n, y(t_n;r)), p_{2,2}(t_n, y(t_n;r)) \right\}$$
$$k_{3,2}(t_n, y(t_n;r)) = \max h\left\{ f(t_n + \frac{h}{4}, u) / u \in (p_{2,1}(t_n, y(t_n;r)), p_{2,2}(t_n, y(t_n;r)) \right\}$$
$$k_{4,1}(t_n, y(t_n;r)) = \min h\left\{ f(t_n + h, u) / u \in (p_{3,1}(t_n, y(t_n;r)), p_{3,2}(t_n, y(t_n;r)) \right\}$$
$$k_{4,2}(t_n, y(t_n;r)) = \max h\left\{ f(t_n + h, u) / u \in (p_{3,1}(t_n, y(t_n;r)), p_{3,2}(t_n, y(t_n;r)) \right\} \qquad (4.4)$$

Where in the fourth order Runge-Kuuta method based on linear combination of Arithmetic mean Harmonic mean and Geometric mean is as follows

$$p_{1,1}(t_n, y(t_n;r)) = \underline{y}(t_n;r) + \frac{h}{4} k_{1,1}(t_n, y(t_n;r))$$
$$p_{1,2}(t_n, y(t_n;r)) = \overline{y}(t_n;r) + \frac{h}{4} k_{1,2}(t_n, y(t_n;r))$$
$$p_{2,1}(t_n, y(t_n;r)) = \underline{y}(t_n;r) + \frac{h}{32}(-k_{1,1}(t_n, y(t_n;r)) + 9k_{1,1}(t_n, y(t_n;r))$$
$$p_{2,2}(t_n, y(t_n;r)) = \overline{y}(t_n;r) + \frac{h}{32}(-k_{1,2}(t_n, y(t_n;r)) + 9k_{1,2}(t_n, y(t_n;r)) \qquad (4.5)$$
$$p_{3,1}(t_n, y(t_n;r)) = \underline{y}(t_n;r) + \frac{h}{96}(-6k_{1,1}(t_n, y(t_n;r)) + 10k_{2,1}(t_n, y(t_n;r)) + 44k_{3,1}(t_n, y(t_n;r)))$$
$$p_{3,2}(t_n, y(t_n;r)) = \overline{y}(t_n;r) + \frac{h}{96}(-6k_{1,2}(t_n, y(t_n;r)) + 10k_{2,2}(t_n, y(t_n;r)) + 44k_{3,2}(t_n, y(t_n;r)))$$

Define

$$\underline{F}(t, y(t;r)) = \left[ \frac{7(\underline{k}_1(t,y(t;r)) + 2\underline{k}_2(t,y(t;r)) + 2\underline{k}_3(t,y(t;r)) + \underline{k}_4(t,y(t;r))) - \left( \frac{3\underline{k}_1(t,y(t;r))\underline{k}_2(t,y(t;r))}{\underline{k}_1(t,y(t;r)) + \underline{k}_2(t,y(t;r))} + \frac{3\underline{k}_2(t,y(t;r))\underline{k}_3(t,y(t;r))}{\underline{k}_2(t,y(t;r)) + \underline{k}_3(t,y(t;r))} + \frac{3\underline{k}_3(t,y(t;r))\underline{k}_4(t,y(t;r))}{\underline{k}_3(t,y(t;r)) + \underline{k}_4(t,y(t;r))} \right) +}{48\left(\sqrt{\underline{k}_1(t,y(t;r)) + \underline{k}_2(t,y(t;r))} + \sqrt{\underline{k}_2(t,y(t;r)) + \underline{k}_3(t,y(t;r))} + \sqrt{\underline{k}_3(t,y(t;r)) + \underline{k}_4(t,y(t;r))}\right)} \right]$$

$$\overline{G}(t, y(t;r)) = \left[ \frac{7(\overline{k}_1(t,\overline{y}(t;r)) + 2\overline{k}_2(t,y(t;r)) + 2\overline{k}_3(t,y(t;r)) + \overline{k}_4(t,y(t;r))) - \left( \frac{3\overline{k}_1(t,y(t;r))\overline{k}_2(t,y(t;r))}{\overline{k}_1(t,y(t;r)) + \overline{k}_2(t,y(t;r))} + \frac{3\overline{k}_2(t,y(t;r))\overline{k}_3(t,y(t;r))}{\overline{k}_2(t,y(t;r)) + \overline{k}_3(t,y(t;r))} + \frac{3\overline{k}_3(t,y(t;r))\overline{k}_4(t,y(t;r))}{\overline{k}_3(t,y(t;r)) + \overline{k}_4(t,y(t;r))} \right) +}{48\left(\sqrt{\overline{k}_1(t,y(t;r)) + \overline{k}_2(t,y(t;r))} + \sqrt{\overline{k}_2(t,y(t;r)) + \overline{k}_3(t,y(t;r))} + \sqrt{\overline{k}_3(t,y(t;r)) + \overline{k}_4(t,y(t;r))}\right)} \right] \quad (4.6)$$

The exact and approximate solutions of at $t_n$, $0 \le n \le N$ are denoted by

$$[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)]$$
$$[y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)]$$

respectively.

By (4.1) and (4.5) we get the exact value as

$$\underline{Y}(t_{n+1};r) \approx \underline{Y}(t_n;r) + \frac{h}{2}\underline{F}[t_n, Y(t_n;r)]$$
$$\overline{Y}(t_{n+1};r) \approx \overline{Y}(t_n;r) + \frac{h}{2}\overline{G}[t_n, Y(t_n;r)] \quad (4.7)$$

The approximate solution is given by

$$\underline{y}(t_{n+1};r) \approx \underline{y}(t_n;r) + \frac{h}{2}\underline{F}[t_n, Y(t_n;r)]$$
$$\overline{y}(t_{n+1};r) \approx \overline{y}(t_n;r) + \frac{h}{2}\overline{G}[t_n, Y(t_n;r)] \quad (4.8)$$

## V. NUMERICAL EXAMPLES

***5.1 Example 1***

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = y(t), \; t \in [0,1] \\ y(0) = (0.8 + 0.125r, 1.1 - 0.1r) \end{cases} \quad 0 < r \le 1$$

The exact solution is given by

$$\underline{Y}(t;r) = \underline{y}(t;r)e^t,$$
$$\overline{Y}(t;r) = \overline{y}(t;r)e^t,$$

At t=1 we get, $Y(1; r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e]$

Comparison between the exact solution and approximate solution of the fourth order runge kutta method based on linear combination of arithmetic mean, harmonic mean and geometric mean method for t =0.5 and h = 0.1 is displayed in figure 5.1

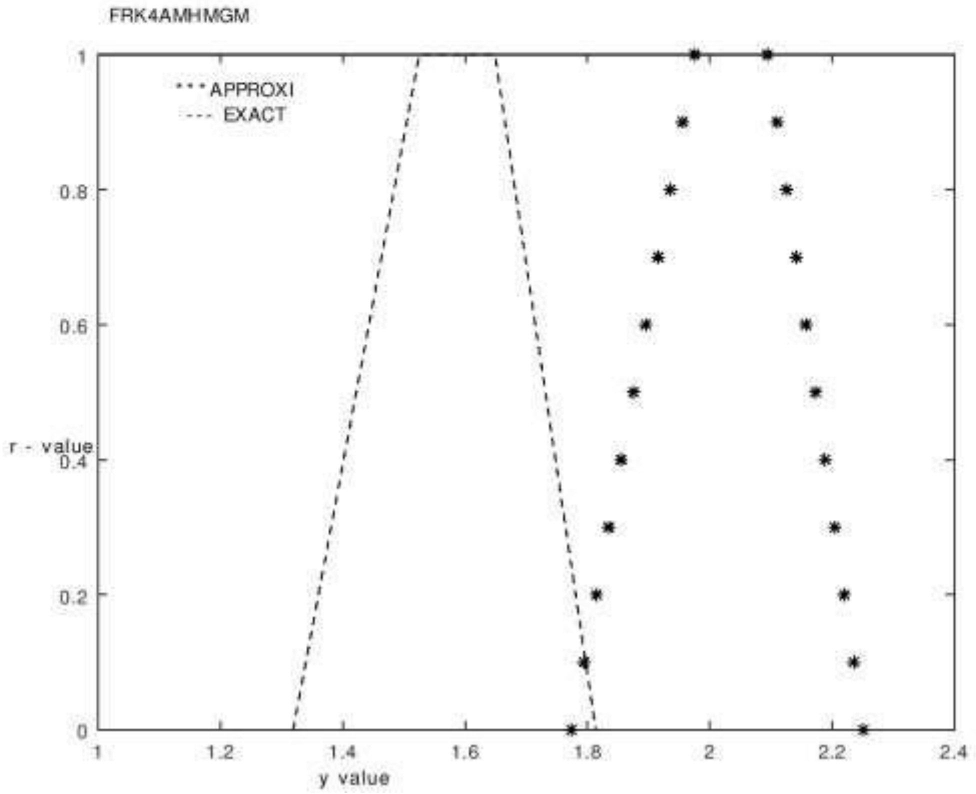


Fig.5.1 Comparison between the exact solution and approximate solution

***5.2 Example 2***

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = 1/y(t),\ t \in [0,1] \\ y(0) = (0.8 + 0.125r, 1.1 - 0.1r) \end{cases} \quad 0 < r \le 1$$

The exact solution is given by

$$\underline{Y}(t;r) = \sqrt{2t + \underline{y}(t_0)}$$
$$\overline{Y}(t;r) = \sqrt{2t + \overline{y}(t_0)}$$

At t=1 we get, $Y(1; r) = (\sqrt{2t + \underline{y}(t_0)}, \sqrt{2t + \overline{y}(t_0)})$

| r | t | $\underline{y}$ | $\overline{y}$ | $\underline{Y}$ | $\overline{Y}$ | Absolute error for lower cut of | Absolute error for upper cut of y |
|---|---|---|---|---|---|---|---|

| | | | | | | y | |
|---|---|---|---|---|---|---|---|
| 0 | 0.5 | 1.774102 | 2.250797 | 1.341641 | 1.449138 | 4.324607e-01 | 8.016594e-01 |
| 0.2 | 0.5 | 1.814677 | 2.219622 | 1.350926 | 1.442221 | 4.637515e-01 | 7.774017e-01 |
| 0.4 | 0.5 | 1.855075 | 2.157044 | 2.188372 | 1.435270 | 4.949284e-01 | 7.531020e-01 |
| 0.6 | 0.5 | 1.895304 | 2.157044 | 1.369306 | 1.428286 | 5.259980e-01 | 7.287585e-01 |
| 0.8 | 0.5 | 1.935371 | 2.125636 | 1.378405 | 1.421267 | 5.569663e-01 | 7.043695e-01 |
| 1 | 0.5 | 1.975283 | 2.094147 | 1.387444 | 1.414214 | 5.878391e-01 | 6.799330e-01 |

Table V.II – Absolute Error Table for Lower cut and Upper cut values

## VI. Conclusions

Solving fuzzy initial value problem has been going on in research using various techniques. A new numerical method is proposed for solving the 1st order fuzzy initial value problem in this paper. The 1st order fuzzy linear differential equation is converted to a fuzzy system and then solved with the fourth order Runge-Kutta method based on linear combination of Arithmetic Mean, Harmonic Mean and Geometric Mean. By comparing the absolute error values at t=0.1 and t=0.5 from the numerical examples 5.1 and 5.2, i.e. from the tables 1 and 2, we find that the proposed method produces good accuracy for solving the FIVPs.

## References


[1]. Abbasbandy.S and Allahviranloo.T, "Numerical solution of fuzzy differential by Runge – Kutta method", *J.Sci.*Teacher Training University, 1(3), 2002.

[2]. Abbasbandy.S, Allahviranloo.T, "Numerical Solution of fuzzy differential equations by Taylor method", *J. of Computational methods in Applied Mathematics 2 (2002)*, 113-124.

[3]. Balachandran. K, Kanagarajan. K, "Existence of solutions of fuzzy delay integro-differential equations with nonlocal condition", *Journal of the Korea Society for Industrial and Applied Mathematics*, 9(2005)65-74.

[4]. Buckley. J.J, T. Feuring, "Fuzzy differential equations", *Fuzzy sets and Systems* 110 (2000) 43-54.

[5]. Bazuaye Frank (2018),"A New 4th order hybrid runge-kutta methods for solving initial value problems (IVPs)" *Pure and Applied Mathematics Journal,7(6):78-87*.

[6]. Bede.B. and Gal, S.G Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equation, *Fuzzy Sets and Systems* 151, 581 – 599, 2005.

[7]. Bede. B., Rudas, I.J. and Bencsik, A.L. "First order linear fuzzy differential equations under generalized differentiability", *Inform.Sci*.177, 1648-1662, 2007.

[8]. Chang.S.L, .Zadeh.L.A, "On Fuzzy mapping and control", *IEEETrans, Systems Man cybernet.2* (1972), 30-34.

[9]. Dubosis.D, Prade.H, "Towards fuzzy differential calculus: Part3, differentiation", *Fuzzy sets and systems 8*(1982), 25-233.

[10]. Ganesan.K and Veeramani.P (2006) "Fuzzy linear programming with trapezoidal fuzzy numbers", *Ann.oper.Res*, 143,305-315

[11]. Goetschel.R, Voxman.Elementry.W, "Fuzzy calculus", *Fuzzy Sets and Systems 18* (1986).31-43.

[12]. Kanagarajan. K, Sambath. M, "Runge- Kutta Nystrom method of order three for solving fuzzy differential equations", *Computational methods in Applied Mathematics*, Vol.10 (2010) , No.2, pp.195-203.

[13]. Kanagarajan. K, Sambath. M, "Numerical solution of fuzzy differential equations by third order Runge-Kutta method", *International journal of Applied Mathematics and Computaion, 2(4)* (2010) pp.1-8.

[14]. Kandel. A, W.J. Byatt, "Fuzzy differential equations", in: *Proceedings of the International Conference on Cybernetics and Society*, Tokyo, (1978) 1213-1216.

[15]. Khattri S.K.(2012),"Euler's Number and Some Means", *Tamsui Oxford Journal of Information and Mathematical Sciences*, pp 369-377.

[16]. Ming Ma, Friedman.M, and Kandel.A, "Numerical solutions of fuzzy differential equations", *Fuzzy Sets and Systems*, 105 (1999),35-48.

[17]. Nirmala. V, Chenthur Pandian. S, "Numerical Solution of Fuzzy Differential Equation by Fourth Order Runge-Kutta Method with Higher Order Derivative Approximations", *European Journal of Scientific Research, Vol.62 N0.2* (2011), pp 198-206.

[18]. Puri.M.L, Ralescu.D.A, "Differentials of Fuzzy functions", *J.Math.Anal.Appl.91* (1983) 552- 558.

[19]. Seikkala. S, "On the fuzzy Initial Value Problem", *Fuzzy Sets and Systems*, 24(1987)319-330.